National Technical University of Athens
SCHOOL OF APPLIED MATHEMATICAL AND PHYSICAL SCIENCES

Poincaré Map Based Contiunuation Methods

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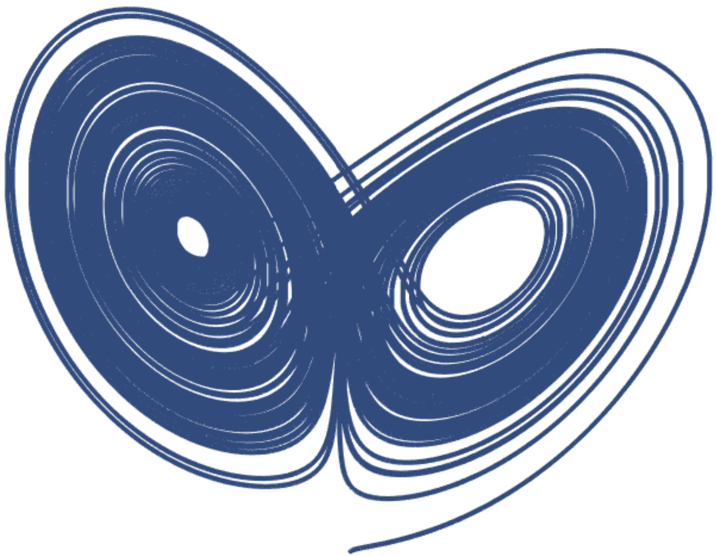
University of Naples Federico II
Department of Structures for Engineering and Architecture

Outline of the Presentation

Constructing Periodic Solutions



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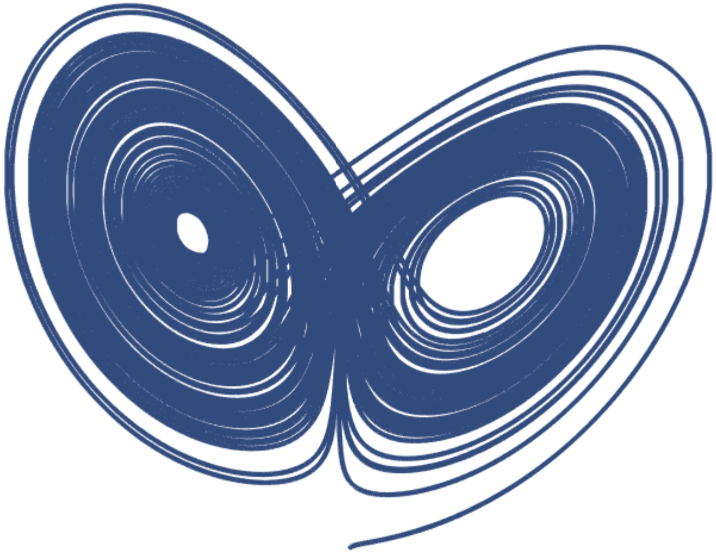

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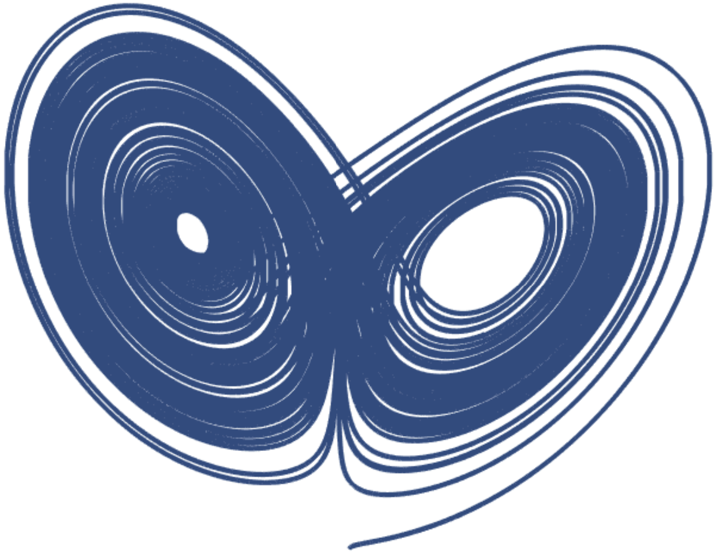

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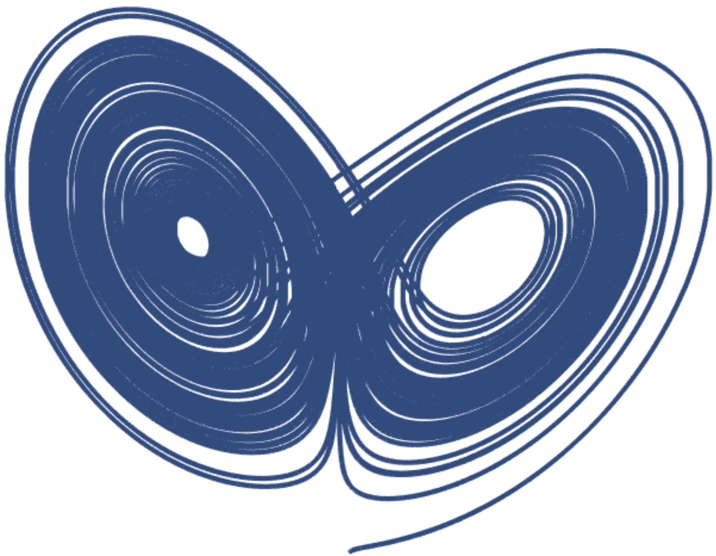

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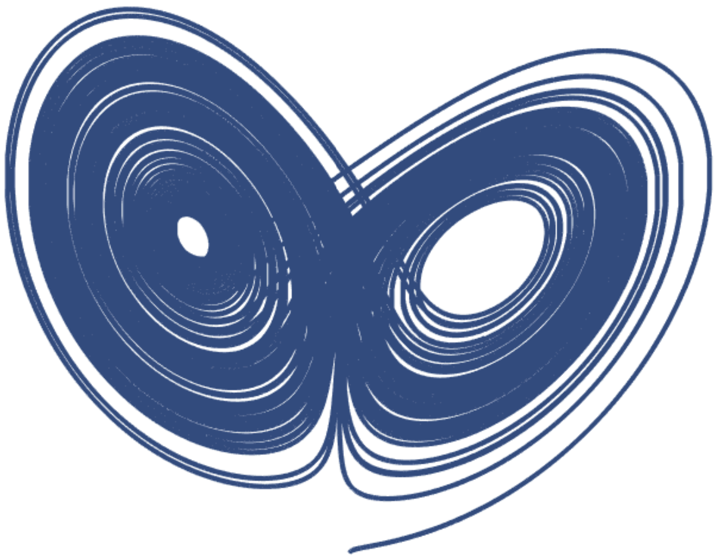

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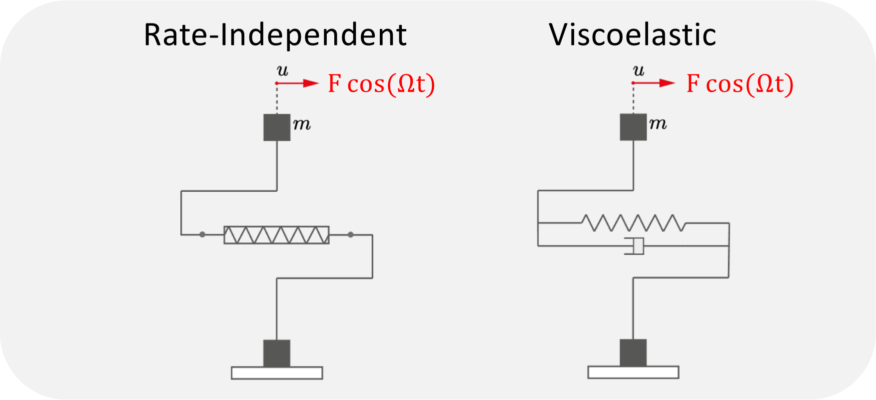

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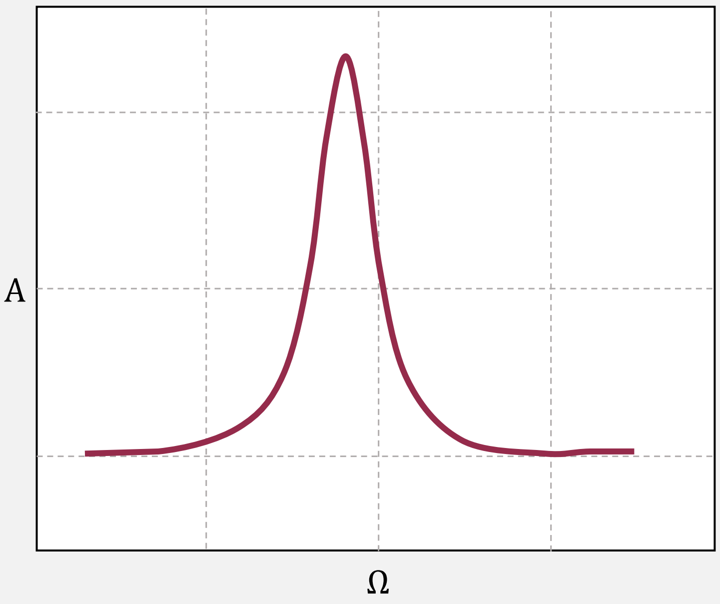

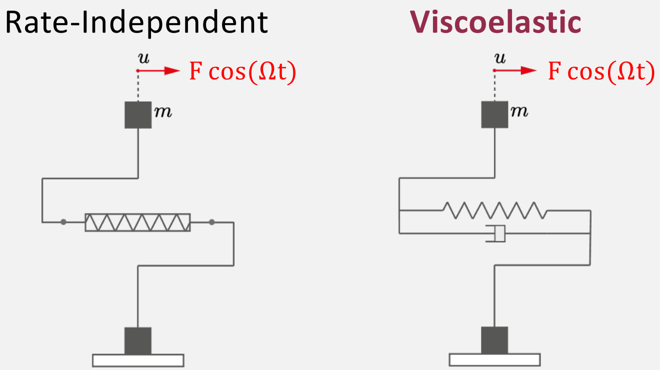

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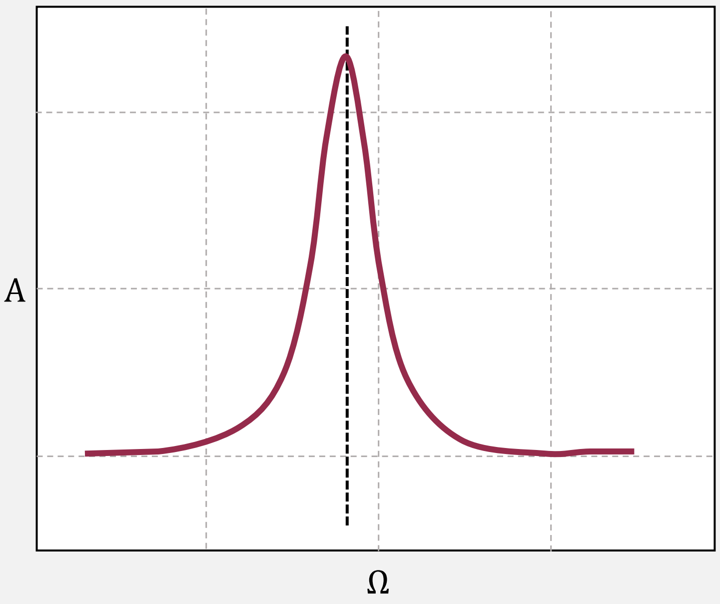

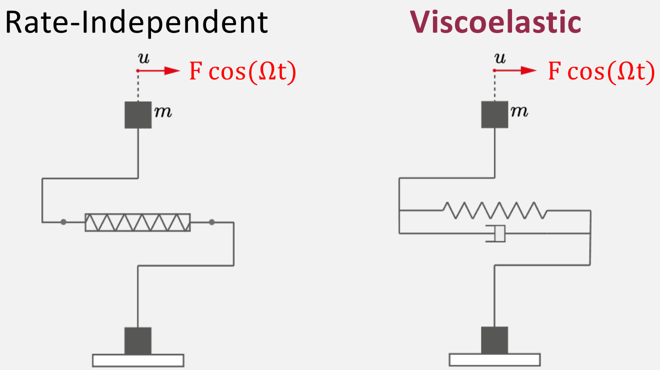

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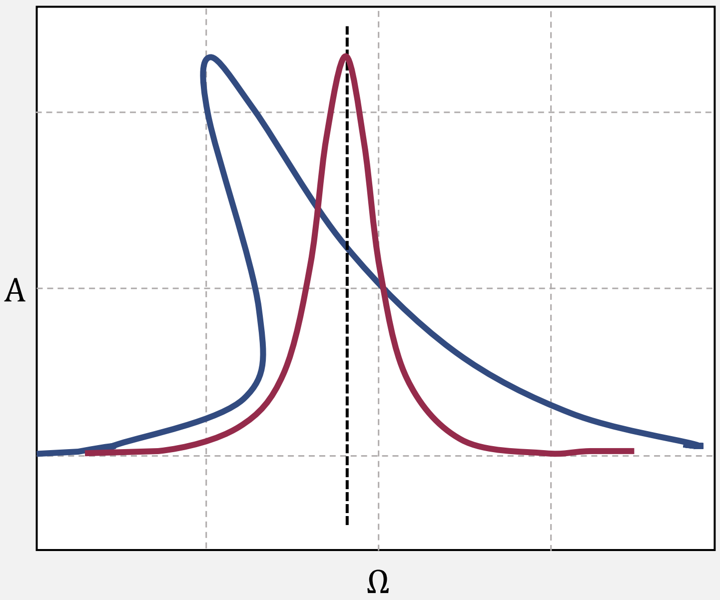

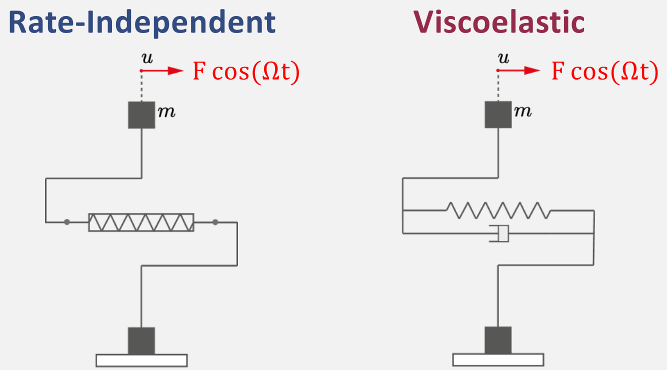

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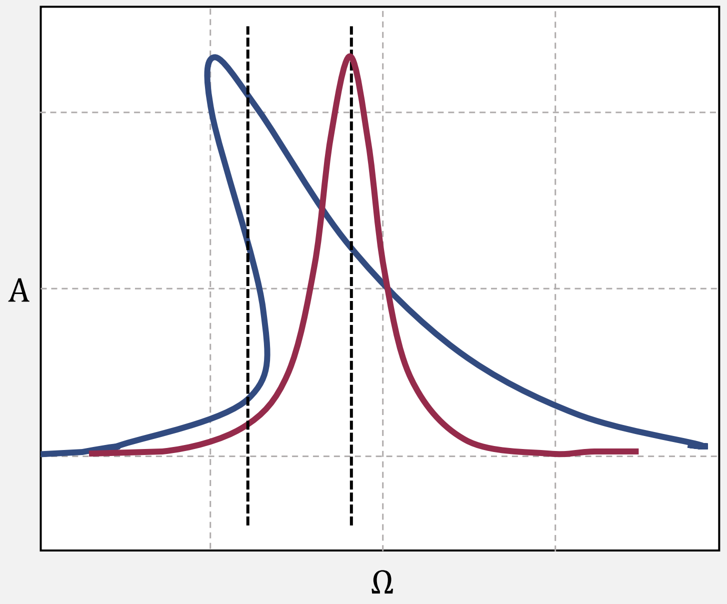

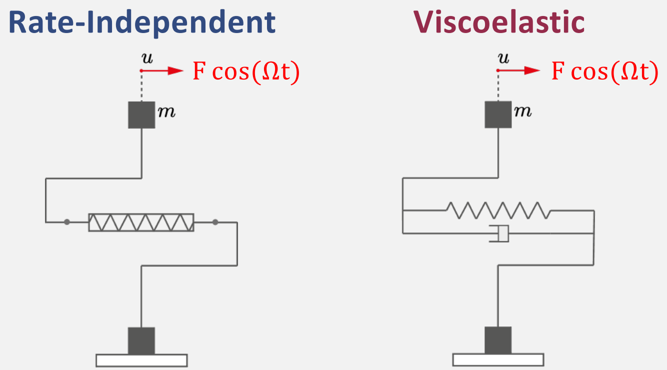

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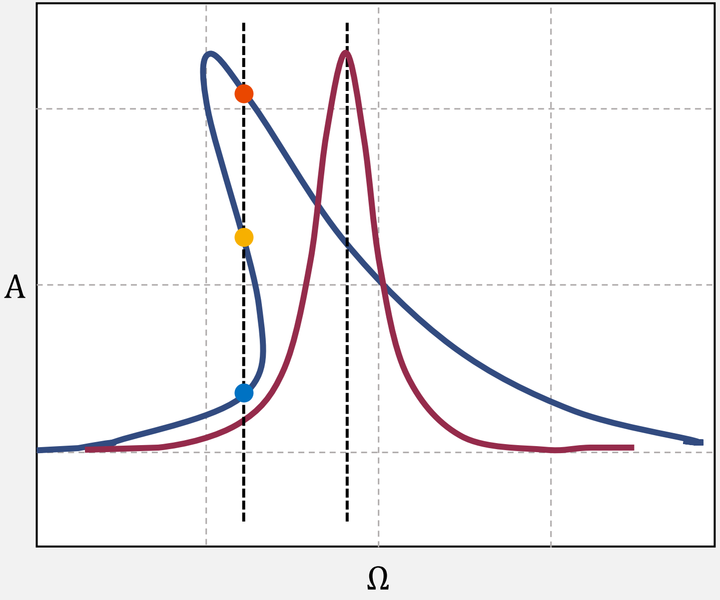
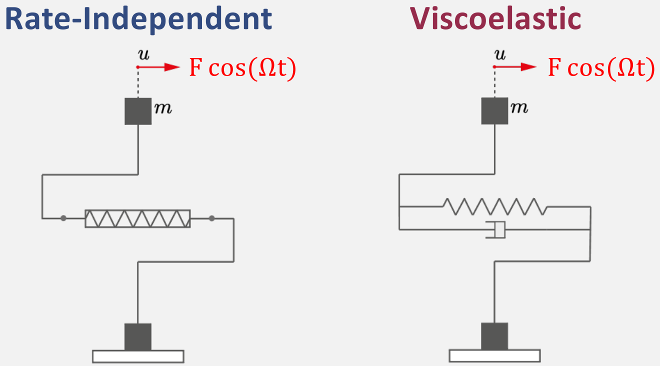
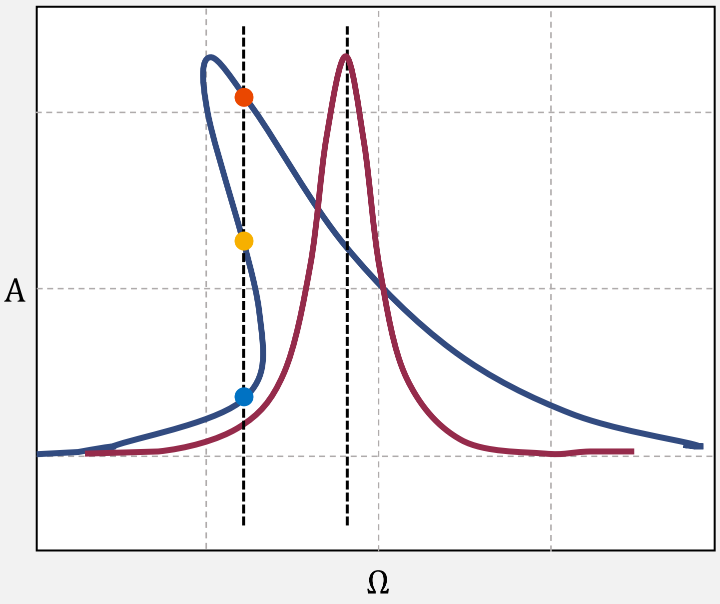
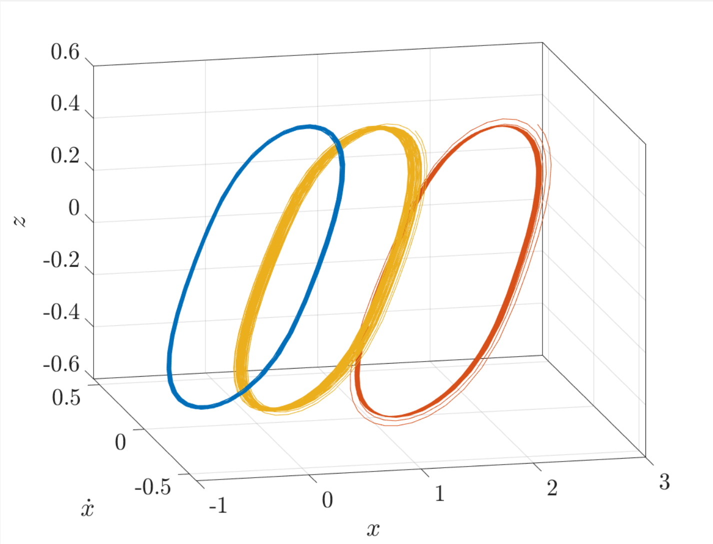




Frequency-Response Curves





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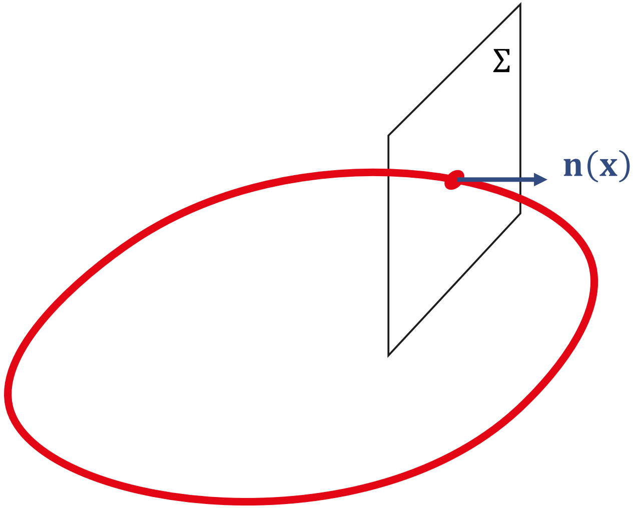

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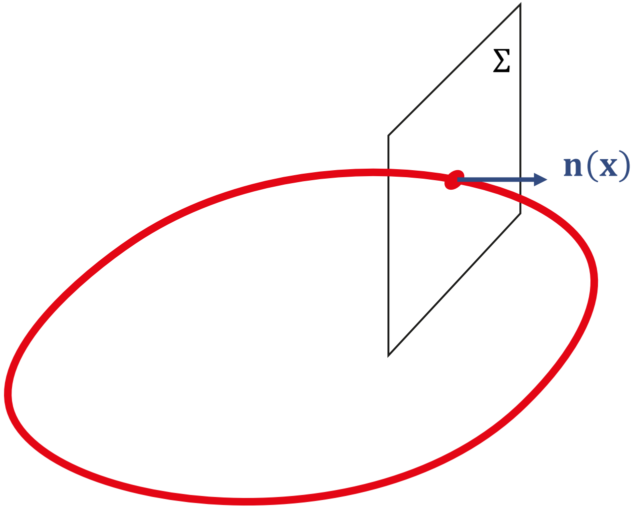

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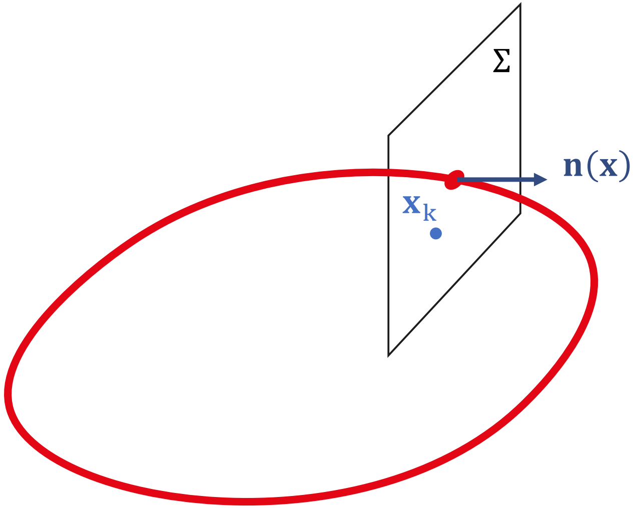

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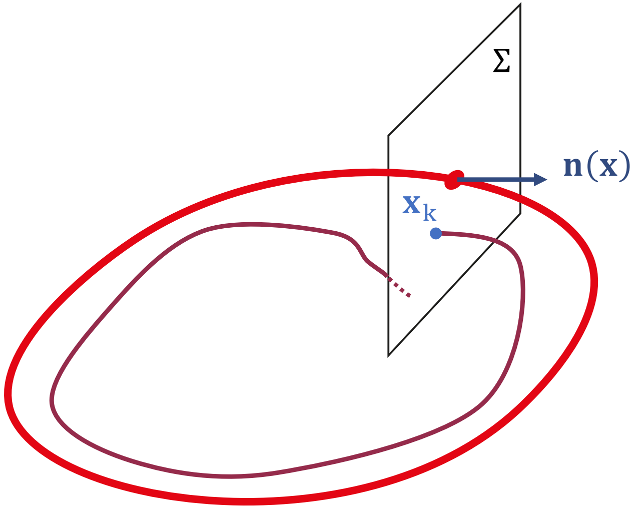

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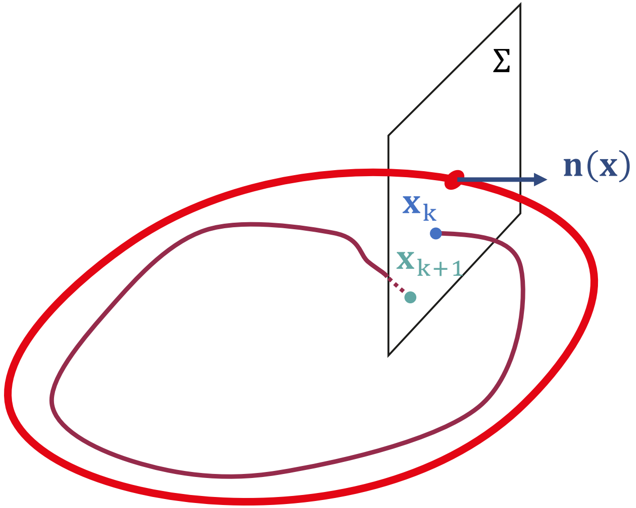

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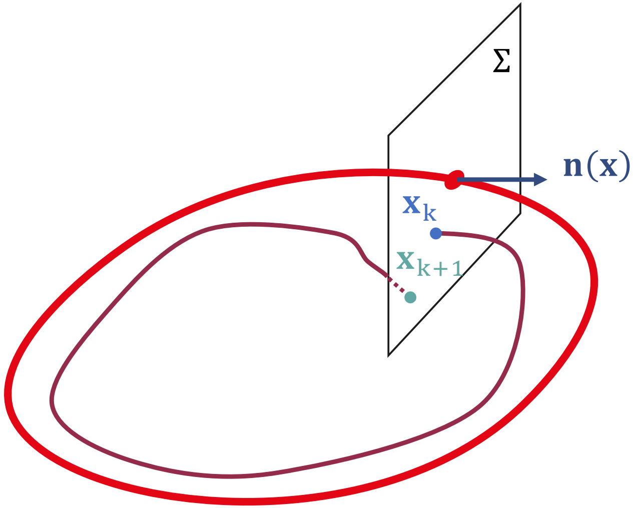

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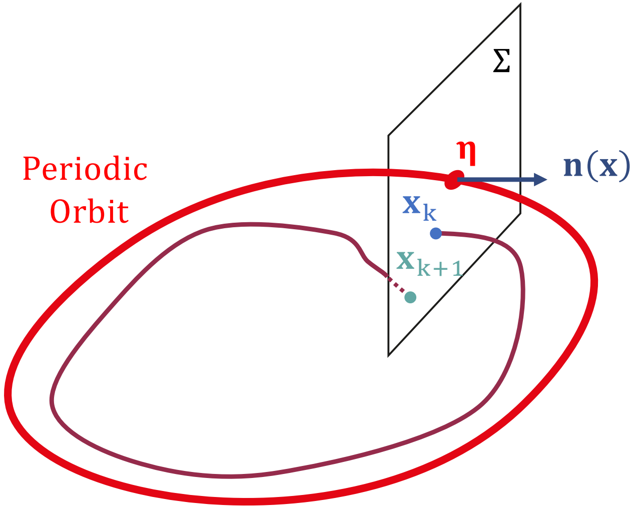
A trajectory that crosses Σ at $\boldsymbol{\eta}$ and comes back to intersect Σ at the same point after an interval of time T is a periodic orbit of period T.



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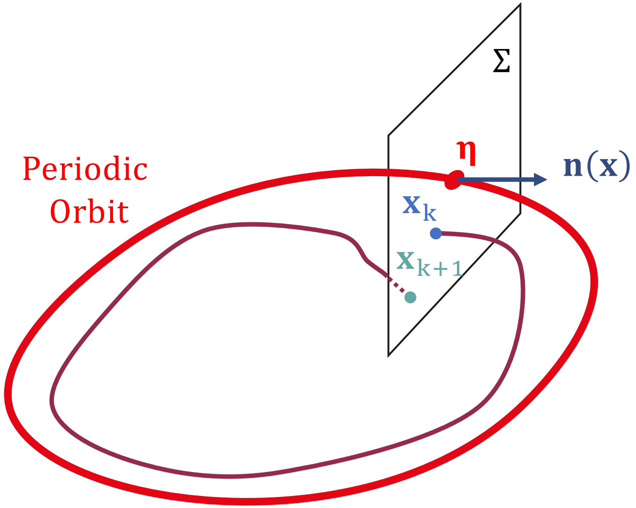

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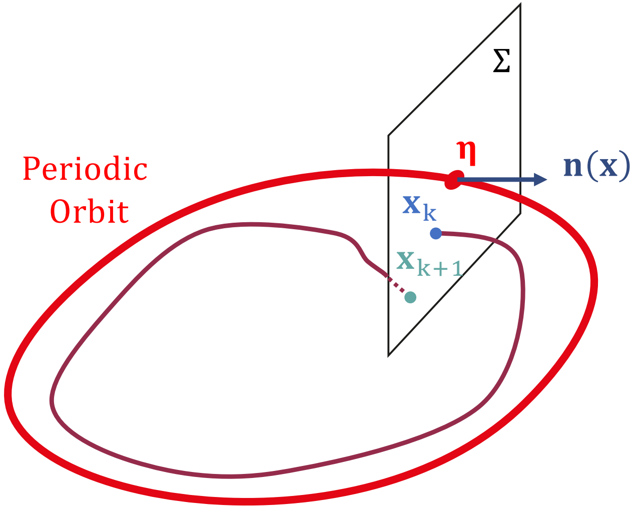

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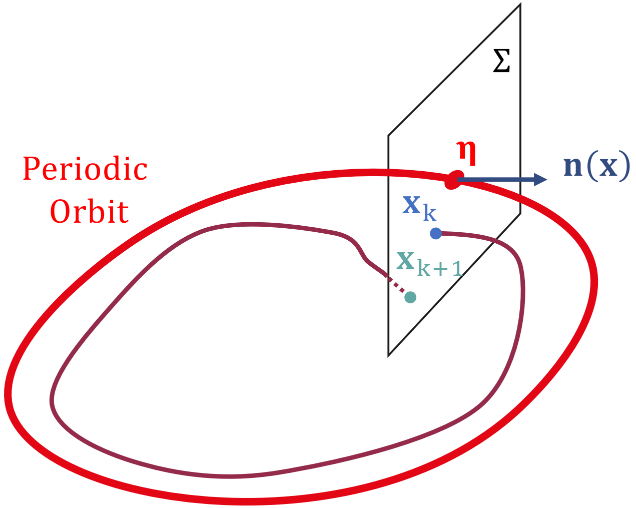

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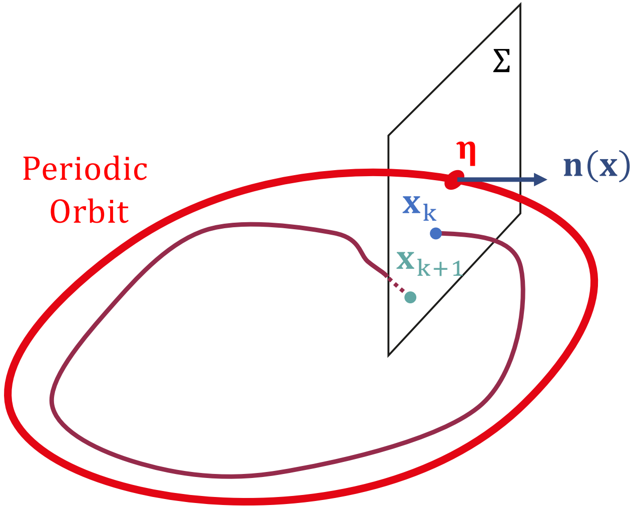

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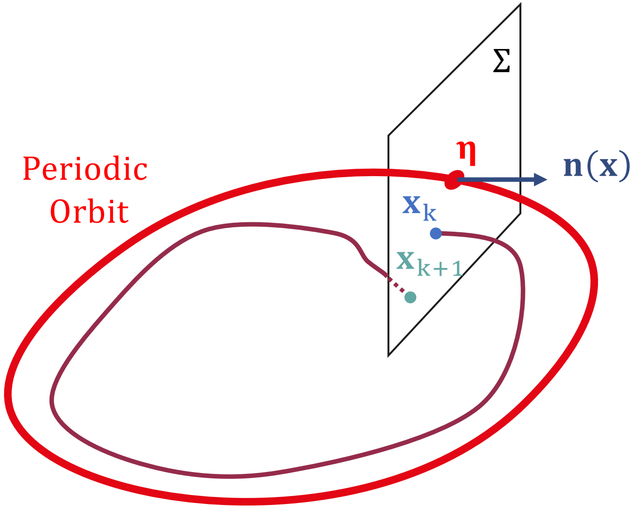
The stability of a periodic solution can be studied through the Poincaré map.



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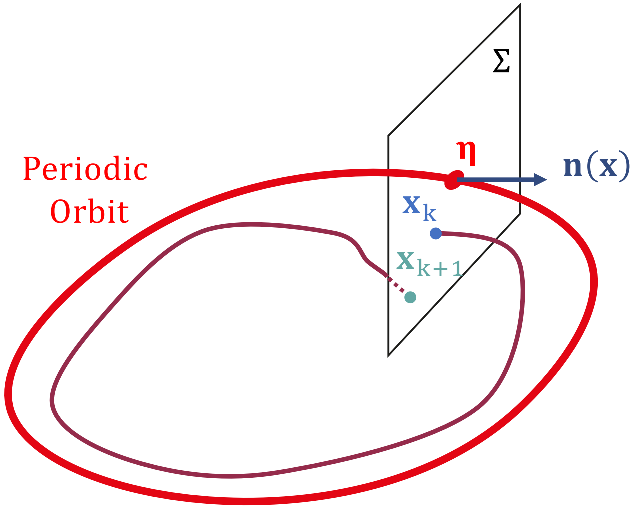

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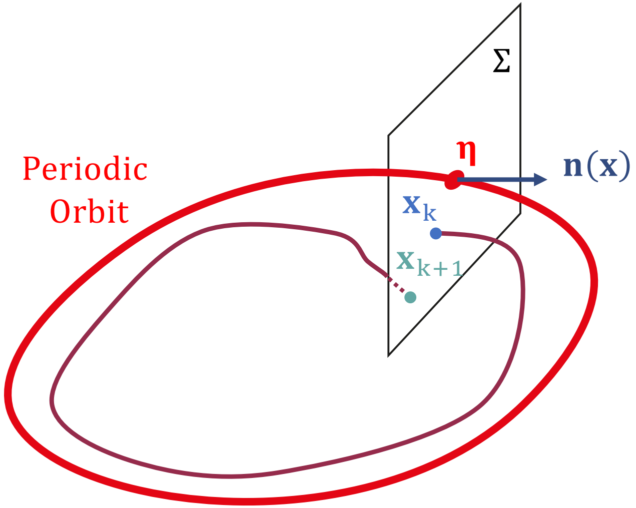

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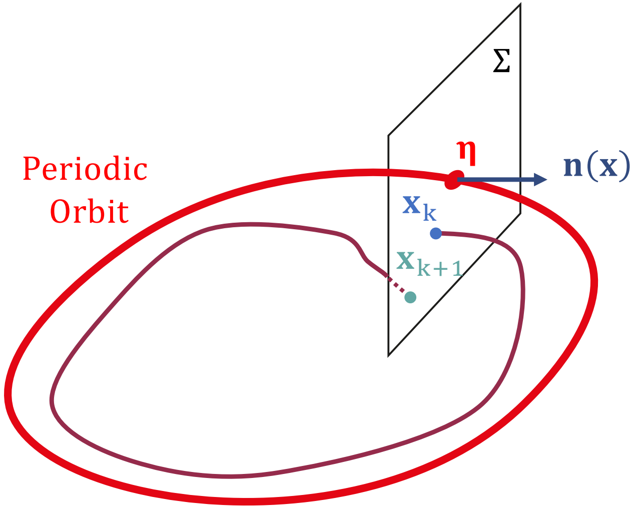

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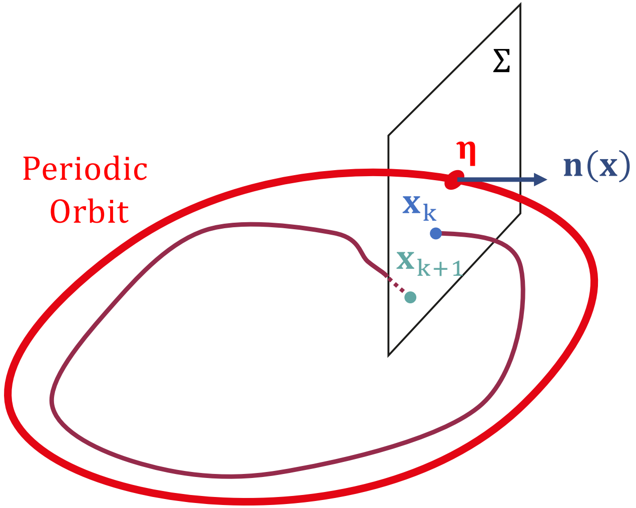

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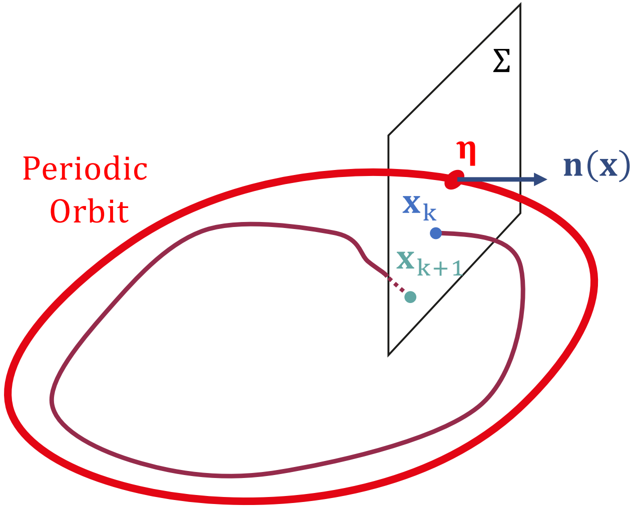

By looking at the behavior of $\mathbf{P}$ near to the fixed point, we can determine the stability of the closed orbit. Thus, the Poincaré map converts problems about closed orbits (which are difficult) into problems about fixed points of a mapping (which are easier in principle, though not always in practice). The snag is that it's typically impossible to find a closed form for $\mathbf{P}$.

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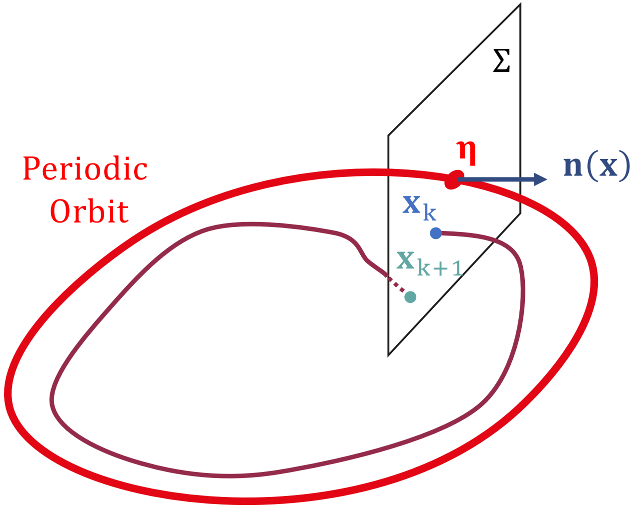

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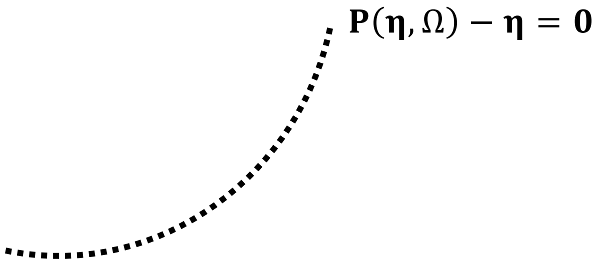

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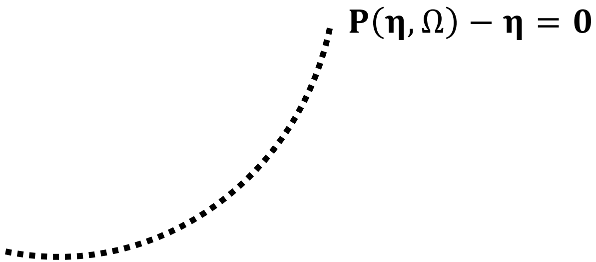

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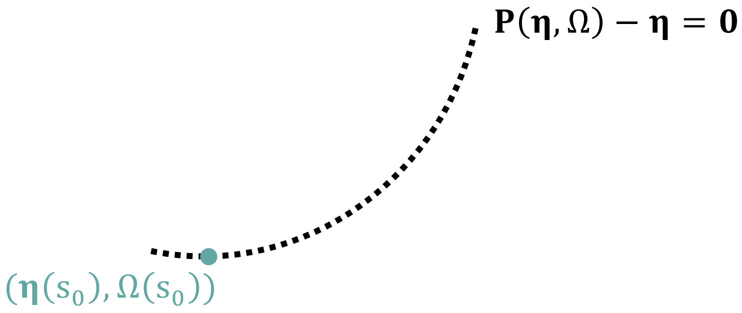

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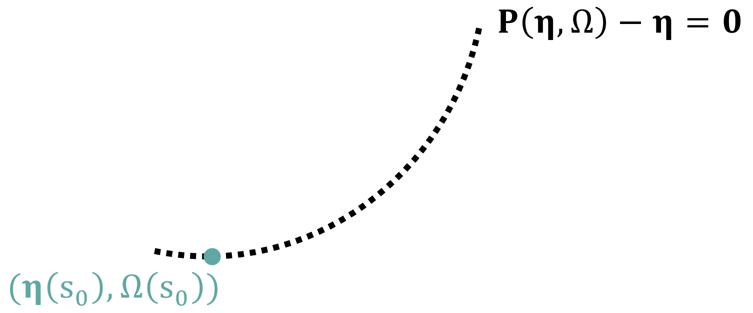
$$\mathbf{P}(\boldsymbol{\eta}, \Omega) - \boldsymbol{\eta} = \mathbf{0}$$

Both the periodic solution $\boldsymbol{\eta}$ and Ω are taken to be function of the arclength s along the solution path:

$$\mathbf{P}[\boldsymbol{\eta}(s), \Omega(s)] - \boldsymbol{\eta}(s) = \mathbf{0}$$

The solution $(\boldsymbol{\eta}(s_0 + \Delta s), \Omega(s_0 + \Delta s))$ is sought as the intersection between the line normal to the unit tangent vector $\mathbf{a}$ passing through the first approximation $(\boldsymbol{\eta}^{(1)}, \Omega^{(1)})$ and the solution curve. Let $\mathbf{b}$ denote the normal vector to the tangent vector $\mathbf{a}$.



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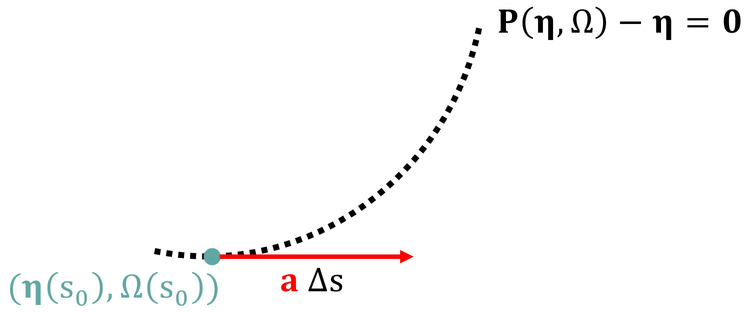

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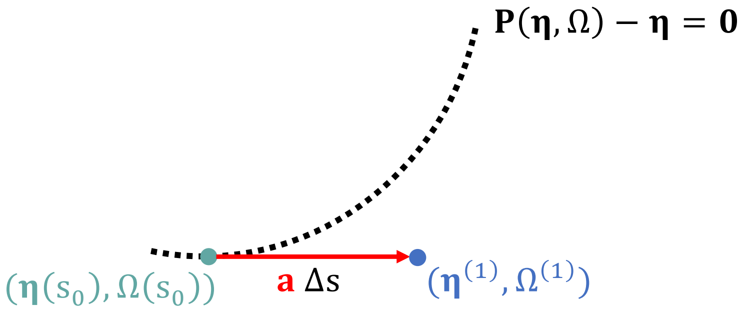

$$\boldsymbol{\eta}^{(1)} = \boldsymbol{\eta}(s_0) + \frac{\partial \boldsymbol{\eta}}{\partial s}\Delta s$$

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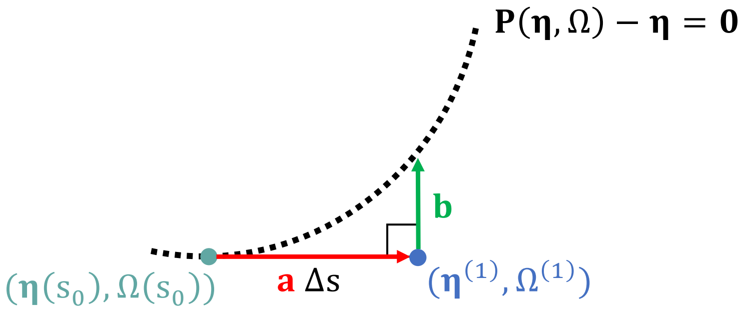

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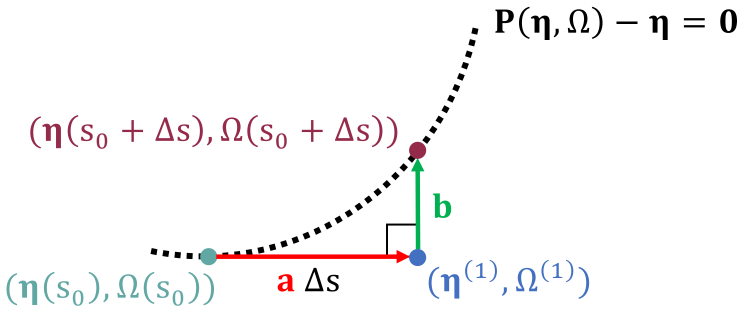

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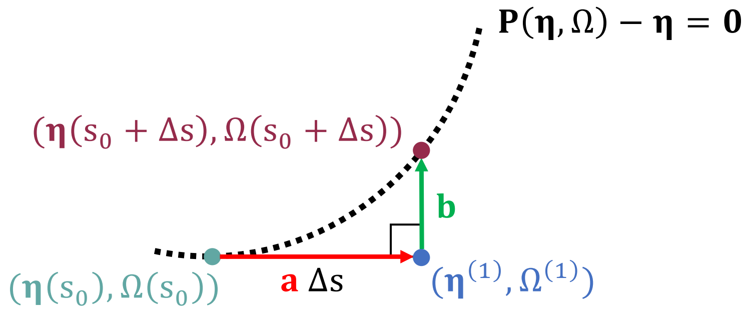
Pseudo-Arclength Pathfollowing of Periodic Solutions

This means that the curve $(\boldsymbol{\eta}, \Omega)$ is sought as the solution of $\mathbf{P}(\boldsymbol{\eta}, \Omega) - \boldsymbol{\eta} = \mathbf{0}$ subject to the following constraint equation $\mathbf{b} \cdot \mathbf{a} = 0$.



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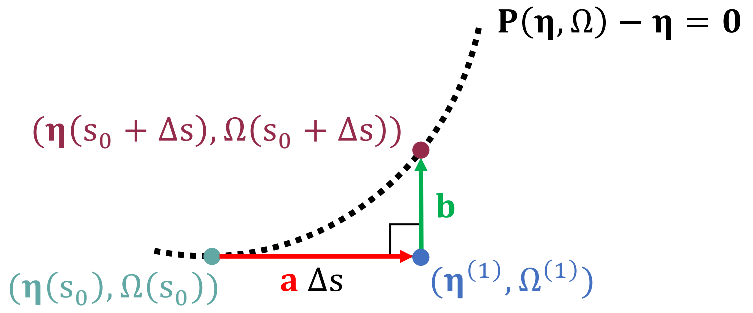

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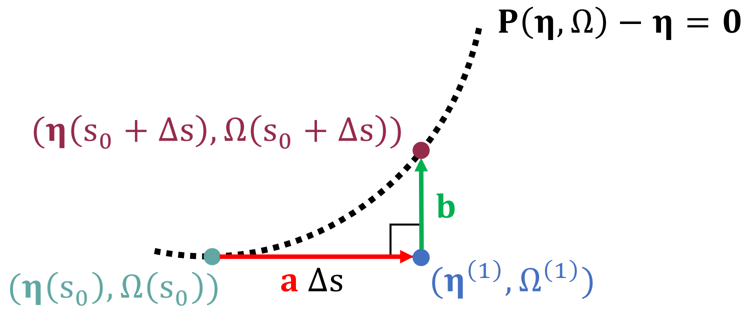
$$g(\boldsymbol{\eta}, \Omega) = 0$$



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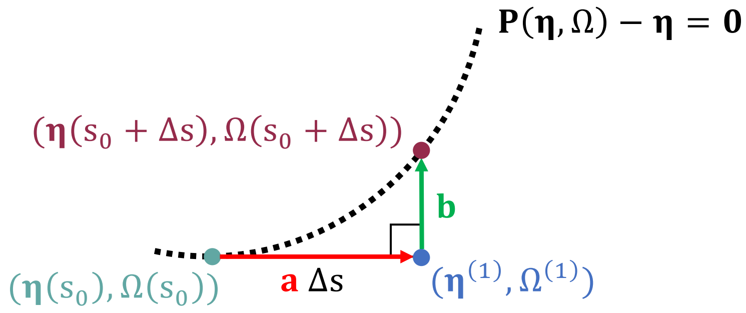
So once we know the first equilibrium point $(\boldsymbol{\eta}(s_0), \Omega(s_0))$ and the increment Δs (which can be made adaptive) the equilibrium path is obtained by:



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So once we know the first equilibrium point $(\boldsymbol{\eta}(s_0), \Omega(s_0))$ and the increment Δs (which can be made adaptive) the equilibrium path is obtained by:

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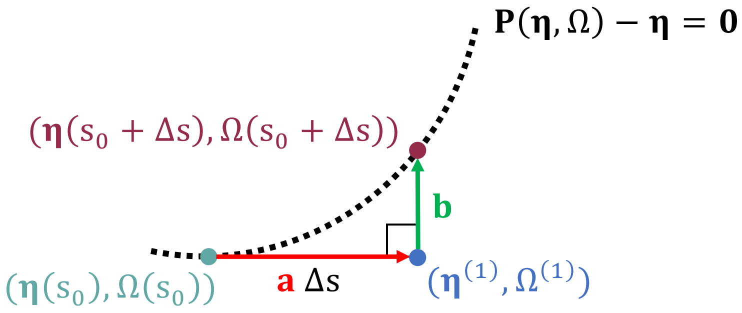


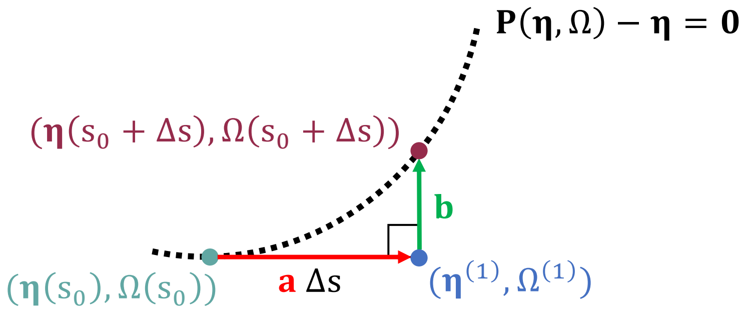
Pseudo-Arclength Pathfollowing of Periodic Solutions

At the j[th] iteration:

$$\boldsymbol{\eta}^{(j)} = \boldsymbol{\eta}^{(j-1)} + \Delta\boldsymbol{\eta}^{(j)} \qquad \Omega^{(j)} = \Omega^{(j-1)} + \Delta\Omega^{(j)}$$

$$\begin{cases} \mathbf{P}\left(\boldsymbol{\eta}^{(j)}, \Omega^{(j)}\right) - \boldsymbol{\eta}^{(j)} = \mathbf{P}\left(\boldsymbol{\eta}^{(j-1)}, \Omega^{(j-1)}\right) - \boldsymbol{\eta}^{(j-1)} + \left[\dfrac{\partial \mathbf{P}}{\partial \boldsymbol{\eta}} - \mathbf{I}\right] \cdot \Delta\boldsymbol{\eta}^{(j)} + \left[\dfrac{\partial \mathbf{P}}{\partial \Omega}\right] \Delta\Omega^{(j)} \\ g\left(\boldsymbol{\eta}^{(j)}, \Omega^{(j)}\right) = g\left(\boldsymbol{\eta}^{(j-1)}, \Omega^{(j-1)}\right) + \dfrac{\partial g}{\partial \boldsymbol{\eta}} \cdot \Delta\boldsymbol{\eta}^{(j)} + \dfrac{\partial g}{\partial \Omega} \Delta\Omega^{(j)} \end{cases}$$

$$\begin{bmatrix} \dfrac{\partial \mathbf{P}}{\partial \boldsymbol{\eta}} - \mathbf{I} & \dfrac{\partial \mathbf{P}}{\partial \Omega} \\ \dfrac{\partial \boldsymbol{\eta}}{\partial s} & \dfrac{\partial \Omega}{\partial s} \end{bmatrix} \begin{bmatrix} \Delta\boldsymbol{\eta}^{(j)} \\ \Delta\Omega^{(j)} \end{bmatrix} = - \begin{bmatrix} \mathbf{P}\left(\boldsymbol{\eta}^{(j-1)}, \Omega^{(j-1)}\right) - \boldsymbol{\eta}^{(j-1)} \\ g\left(\boldsymbol{\eta}^{(j-1)}, \Omega^{(j-1)}\right) \end{bmatrix}$$



$$\mathbf{J}^{(j-1)} \Delta\mathbf{x}^{(j)} = -\mathbf{r}^{(j-1)}$$

$$\Delta\mathbf{x}^{(j)} = -\left[\mathbf{J}^{(j-1)}\right]^{-1} \mathbf{r}^{(j-1)} \qquad \text{error } \left|\left|\mathbf{r}\right|\right| < \text{tol}$$

Pseudo-Arclength Pathfollowing of Periodic Solutions

$$\begin{bmatrix} \dfrac{\partial \mathbf{P}}{\partial \boldsymbol{\eta}} - \mathbf{I} & \dfrac{\partial \mathbf{P}}{\partial \Omega} \\ \dfrac{\partial \boldsymbol{\eta}}{\partial s} & \dfrac{\partial \Omega}{\partial s} \end{bmatrix} \begin{bmatrix} \Delta \boldsymbol{\eta}^{(j)} \\ \Delta \Omega^{(j)} \end{bmatrix} = - \begin{bmatrix} \mathbf{P}\left(\boldsymbol{\eta}^{(j-1)}, \Omega^{(j-1)}\right) - \boldsymbol{\eta}^{(j-1)} \\ g\left(\boldsymbol{\eta}^{(j-1)}, \Omega^{(j-1)}\right) \end{bmatrix}$$

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$$\frac{\partial \mathbf{P}}{\partial \eta_k} \approx \frac{\mathbf{P}[\boldsymbol{\eta} + \delta_1 \mathbf{e}_k; \Omega] - \mathbf{P}[\boldsymbol{\eta} - \delta_1 \mathbf{e}_k; \Omega]}{2\delta_1}$$

$$\frac{\partial \mathbf{P}}{\partial \Omega} \approx \frac{\mathbf{P}[\boldsymbol{\eta}; \Omega + \delta_2] - \mathbf{P}[\boldsymbol{\eta}; \Omega - \delta_2]}{2\delta_2}$$

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After achieving convergence, the procedure furnishes the Jacobian matrix evaluated at the periodic solution (i.e. the monodromy matrix $\boldsymbol{\Phi}$); that is:

$$\boldsymbol{\Phi} = \frac{\partial \mathbf{P}}{\partial \boldsymbol{\eta}}$$

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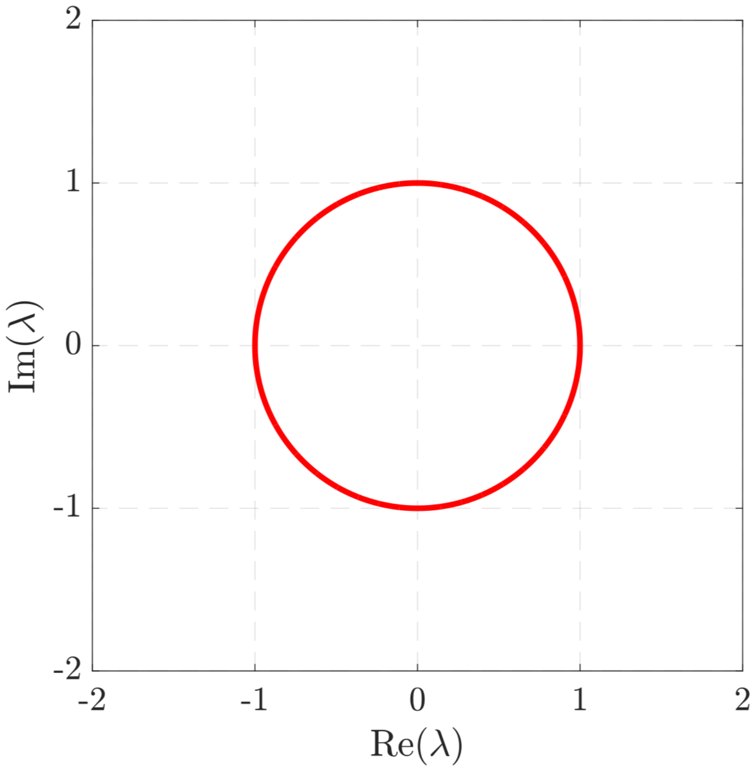
The eigenvalues of $\boldsymbol{\Phi}$, i.e. the Floquet multipliers, allow us to ascertain the stability of the calculated orbit and its bifurcations.

Stability and Bifurcation of Periodic Motion



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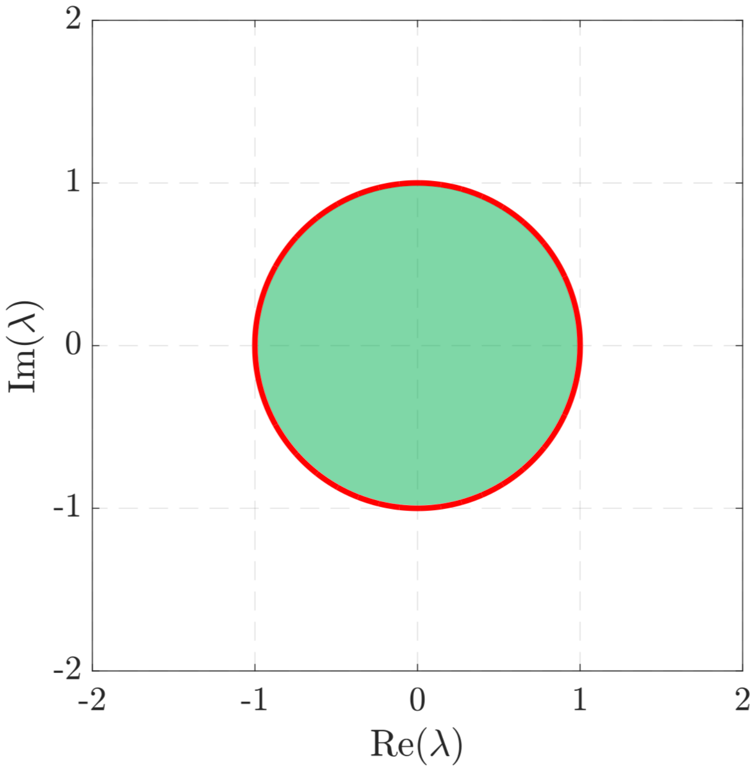

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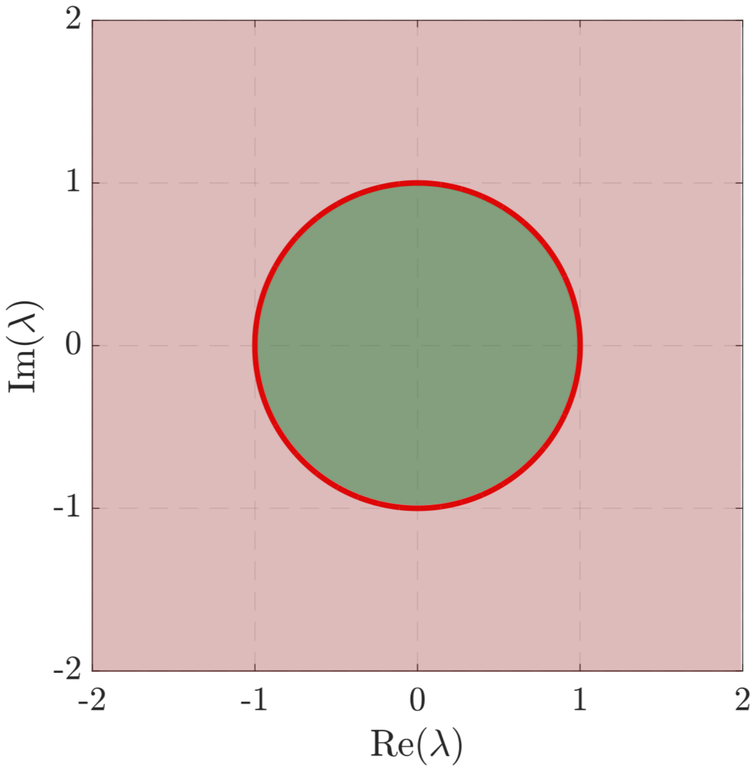

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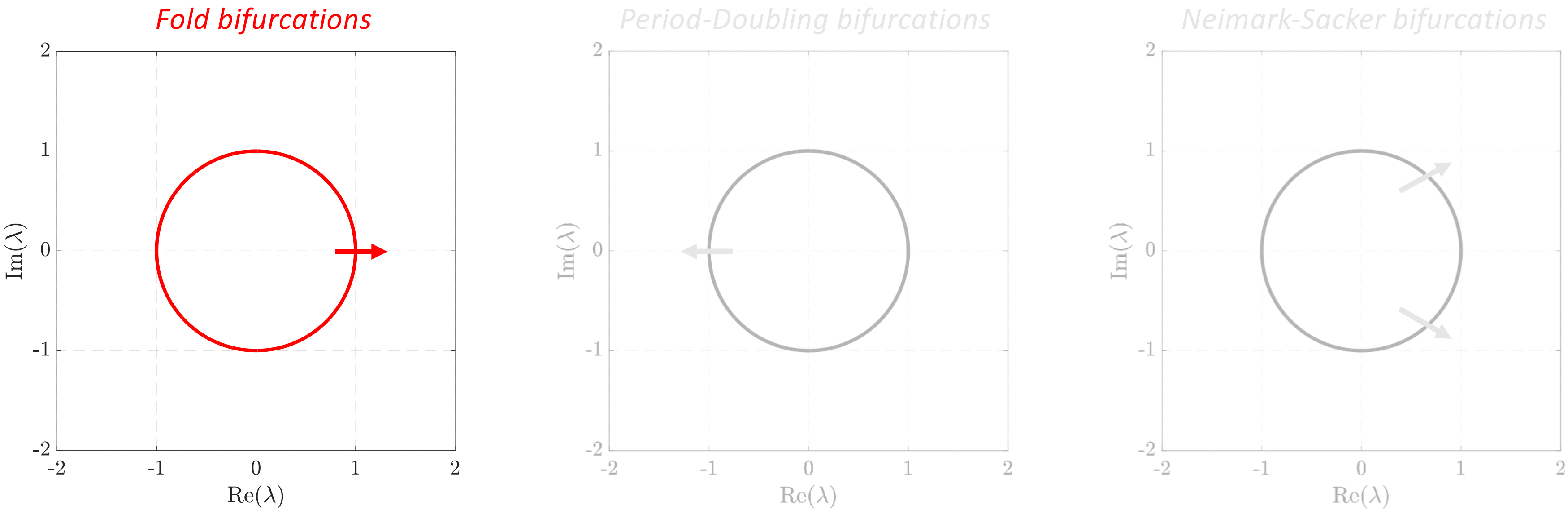

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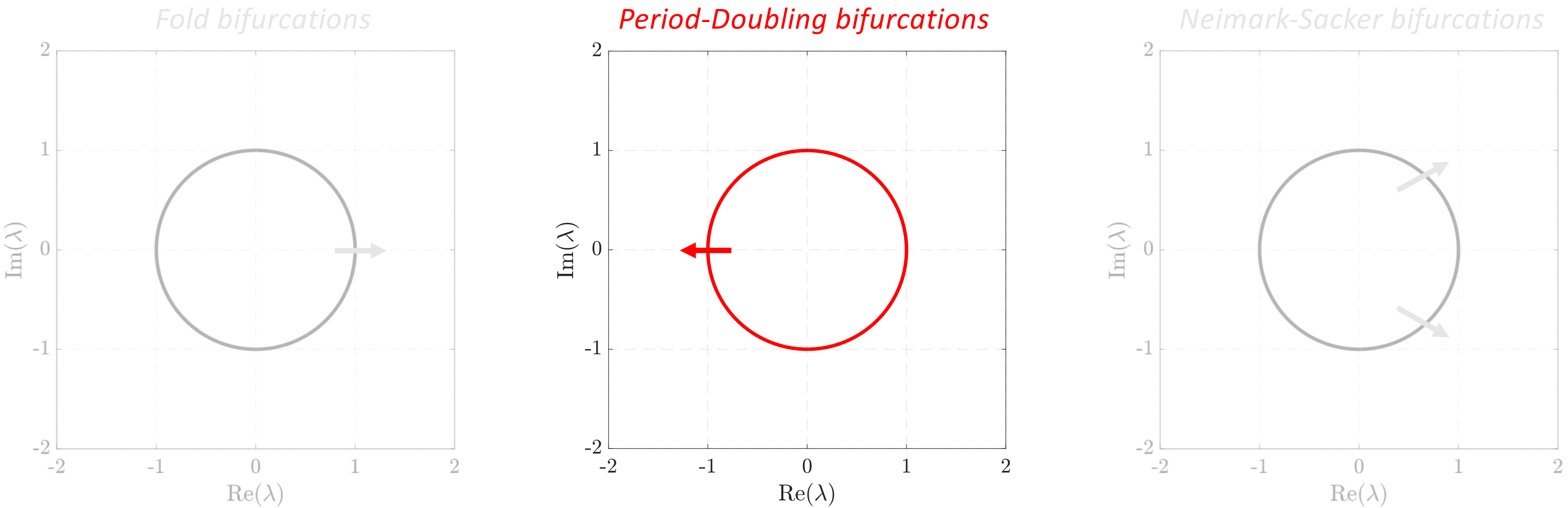

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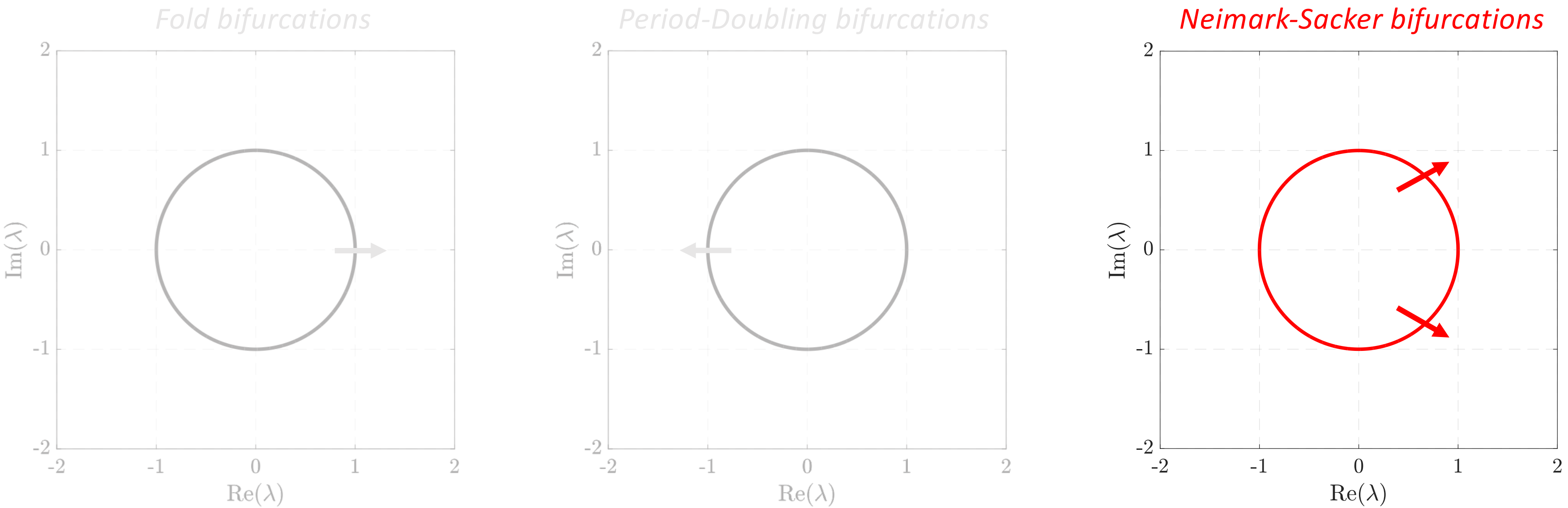
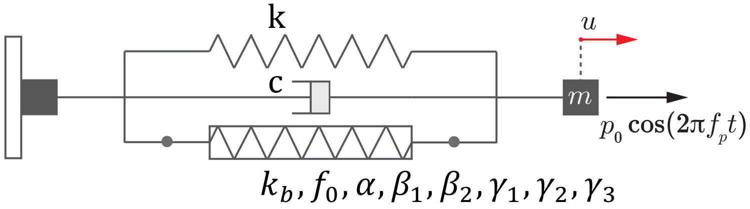


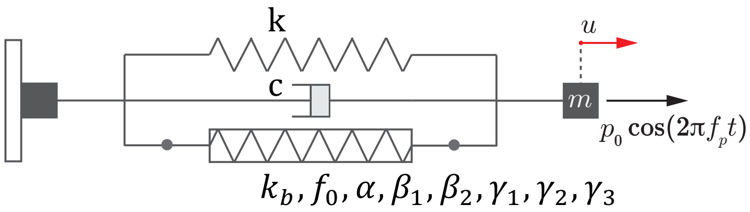
Numerical Applications



$$\mathbf{par} = [\mathrm{m}, \mathrm{k}, \mathrm{c}, k_b, f_0, \alpha, \beta_1, \beta_2, \gamma_1, \gamma_2, \gamma_3, \mathrm{p}_0, \mathrm{f}_\mathrm{p}]$$

$$\begin{cases} m\ddot{u} + c\dot{u} + ku + f = p_0 \cos(2\pi f_p t) \\ \dot{f} = \{k_e(u) + k_b + f_0 + s[f_e(u) + k_b u - f]\}\, \dot{u} \end{cases}$$

Numerical Applications



$$\mathbf{par} = [\mathrm{m}, \mathrm{k}, \mathrm{c}, k_b, f_0, \alpha, \beta_1, \beta_2, \gamma_1, \gamma_2, \gamma_3, \mathrm{p_0}, \mathrm{f_p}]$$

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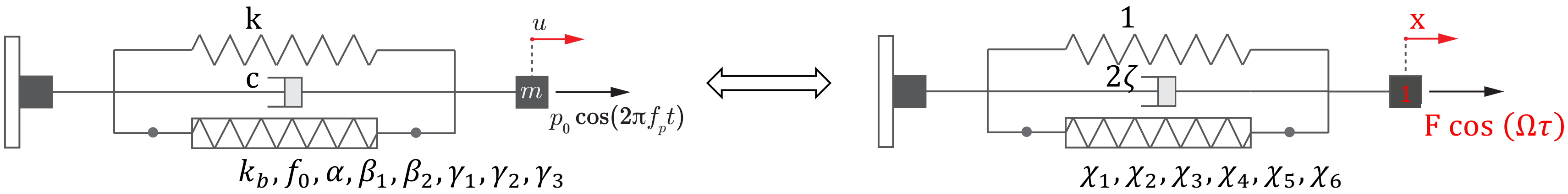
$$u = \frac{1}{\alpha}x \quad f = f_0 z \quad t = \sqrt{\frac{m}{k}}\tau$$

Numerical Applications



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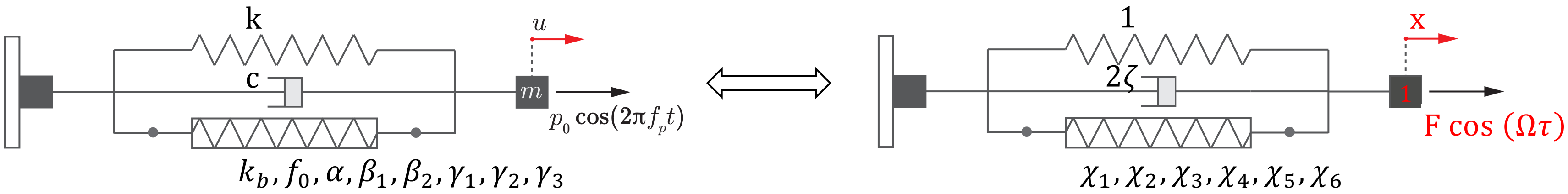
$$\mathbf{par} = [\zeta, \chi_1, \chi_2, \chi_3, \chi_4, \chi_5, \chi_6, F, \Omega] \in \mathbb{R}$$

$$\begin{cases} \dfrac{d^2x}{d\tau^2} + \dfrac{c}{\sqrt{mk}}\dfrac{dx}{d\tau} + x + z = \dfrac{p_0}{f_0}\cos(\Omega\tau) \\ \dot{z} = \{k_e(x) + \chi_6 + 1 + s[f_e(x) + \chi_6 x - z]\}\, \dot{x} \end{cases}$$

$$u = \frac{1}{\alpha}x \qquad f = f_0 z \qquad t = \sqrt{\frac{m}{k}}\tau$$

Numerical Applications



$$\mathbf{par} = [\mathrm{m}, \mathrm{k}, \mathrm{c}, k_b, f_0, \alpha, \beta_1, \beta_2, \gamma_1, \gamma_2, \gamma_3, \mathrm{p}_0, \mathrm{f}_\mathrm{p}]$$

$$\mathbf{par} = [\zeta, \chi_1, \chi_2, \chi_3, \chi_4, \chi_5, \chi_6, \mathrm{F}, \Omega] \in \mathbb{R}$$

Since the two systems are equivalent, we apply the Pseudo-Arclength Path Following method to the nondimensional system. Specifically, the following parameters are fixed:

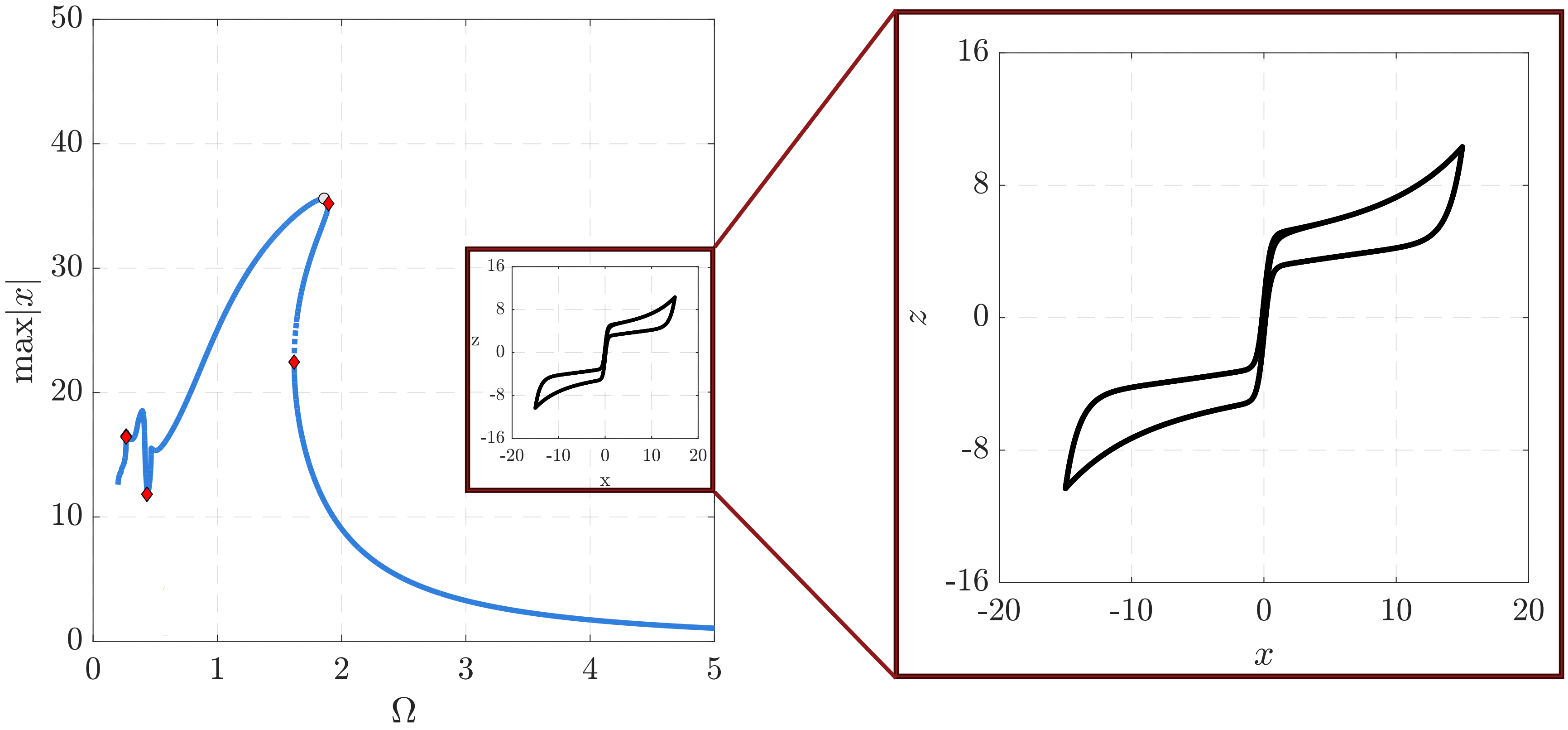
$$\mathbf{par} = [\zeta, \chi_1, \chi_2, \chi_3, \chi_4, \chi_5, \chi_6, \mathrm{F}, \Omega]$$

The frequency of the forcing Ω is assumed as the **control parameter** in the procedure. All the dynamic phenomena observed in the system are evaluated while varying the control parameter (see codimension-1 bifurcation).

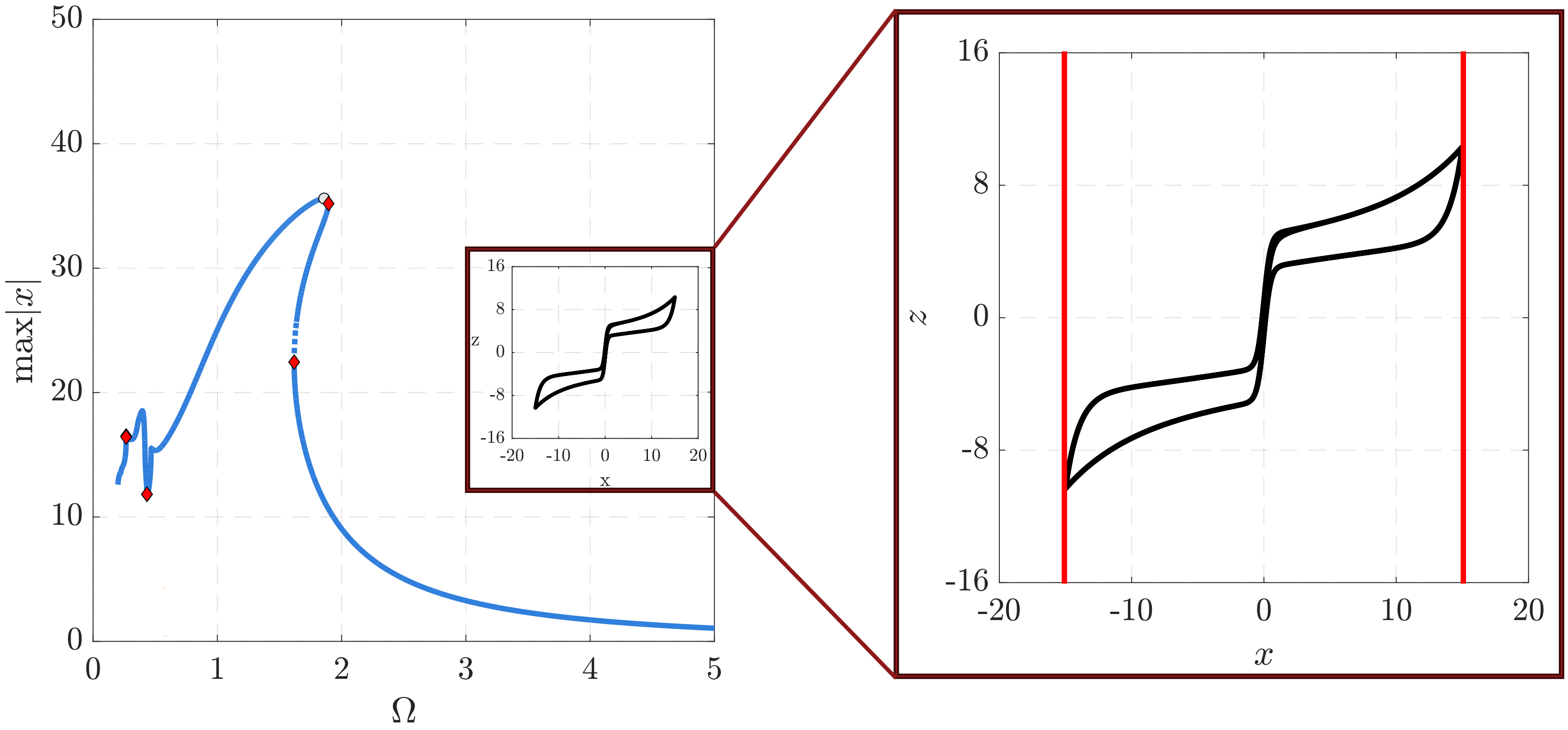
$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \Omega)$$

The procedure provides, as the control parameter varies, the vectors in the state space on a limit cycle.

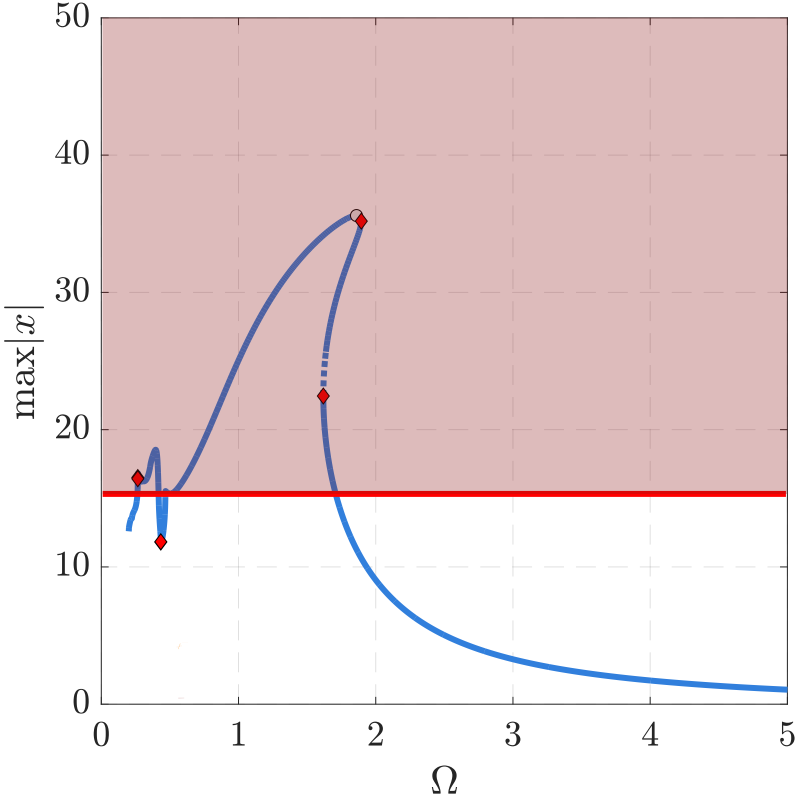
Numerical Applications

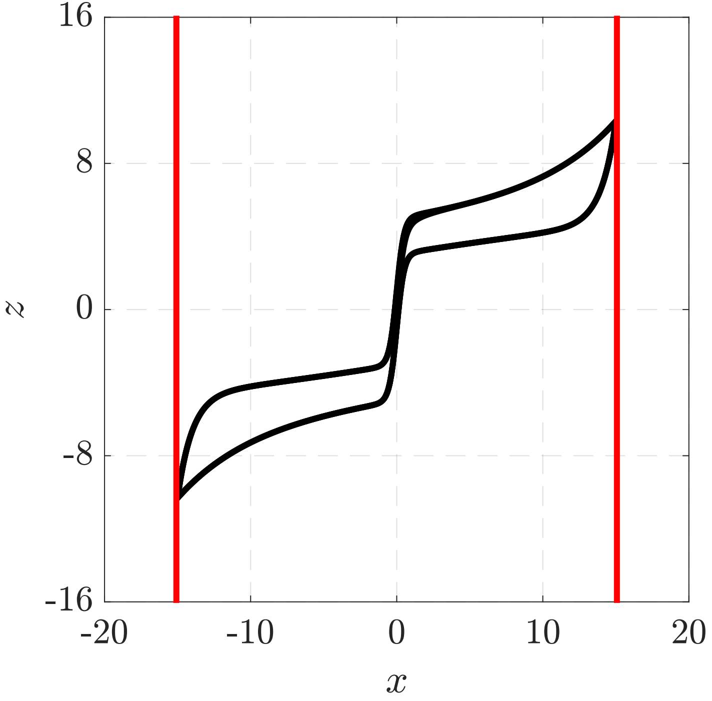


Numerical Applications



Numerical Applications





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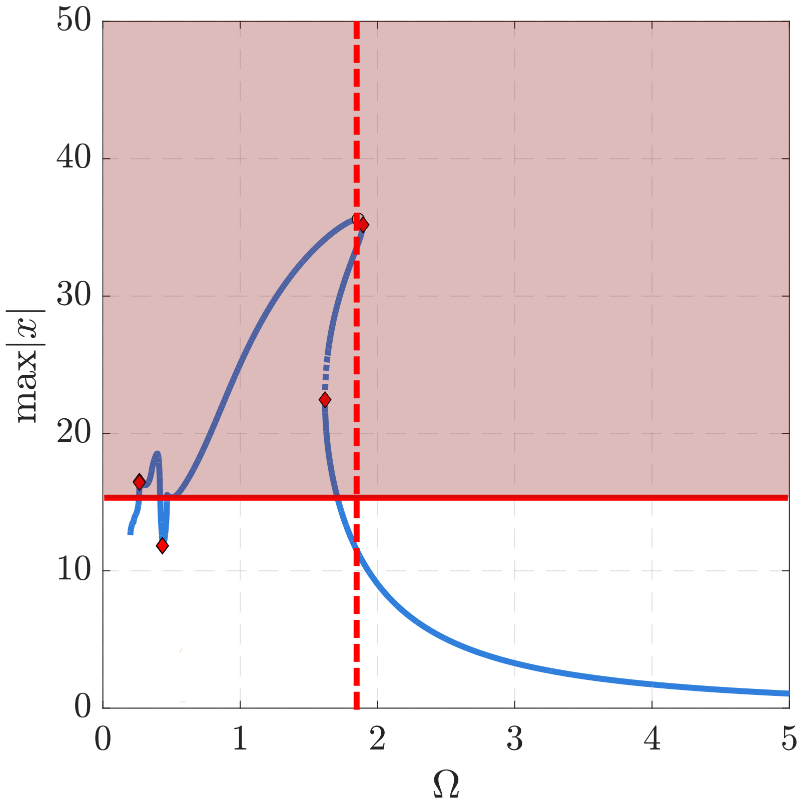
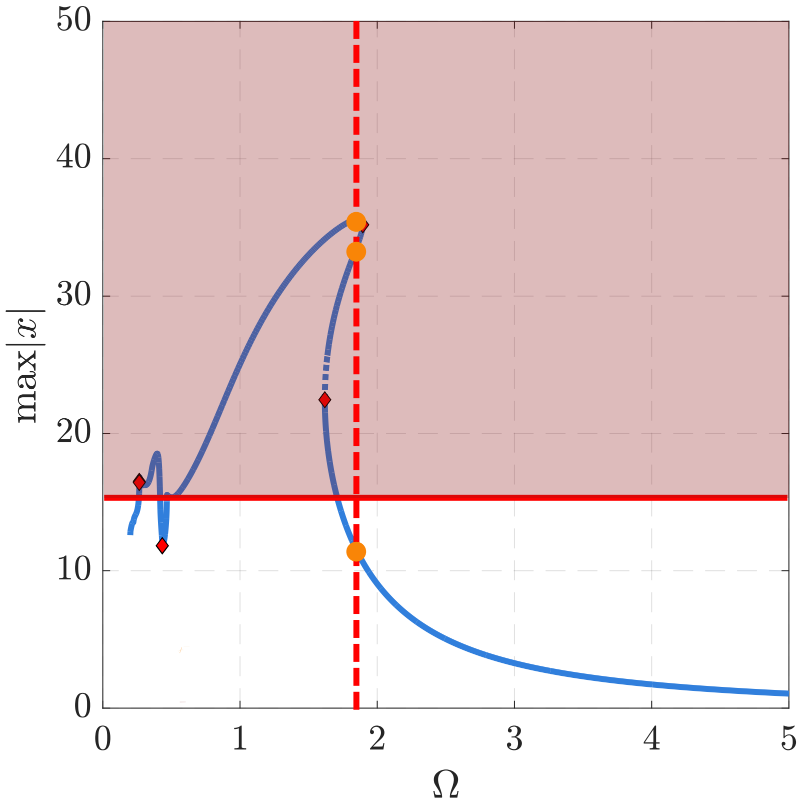


This means that if we excite the system under consideration with a dimensionless forcing of the type $F\cos(\Omega\tau)$

where $F = 20$ and $\Omega = 1.859$ the system will exhibit:

Numerical Applications

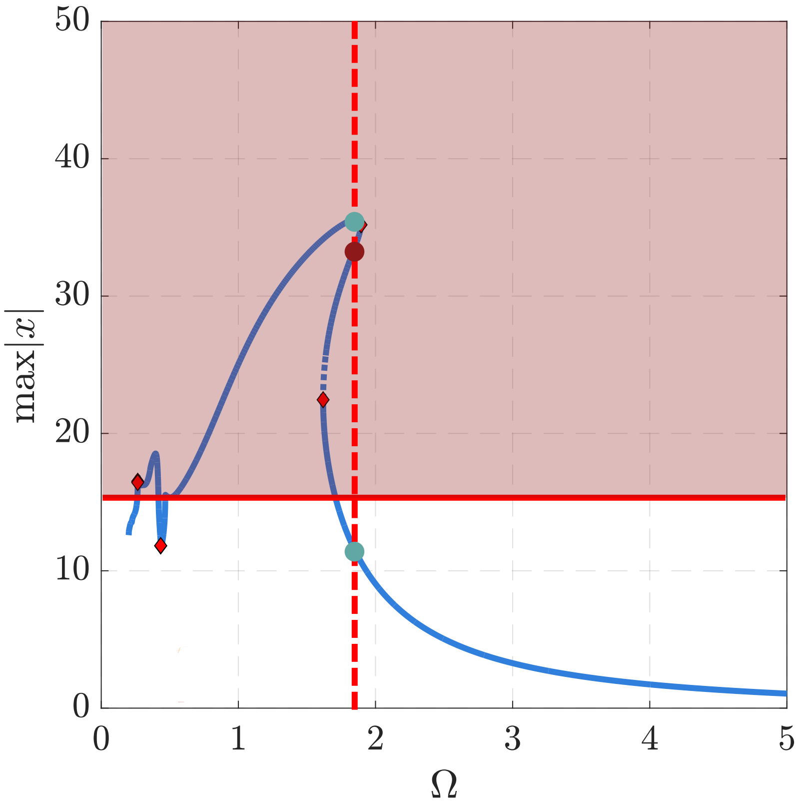


This means that if we excite the system under consideration with a dimensionless forcing of the type $F\cos(\Omega\tau)$

where $F = 20$ and $\Omega = 1.859$ the system will exhibit:

- 3 periodic orbits

Numerical Applications

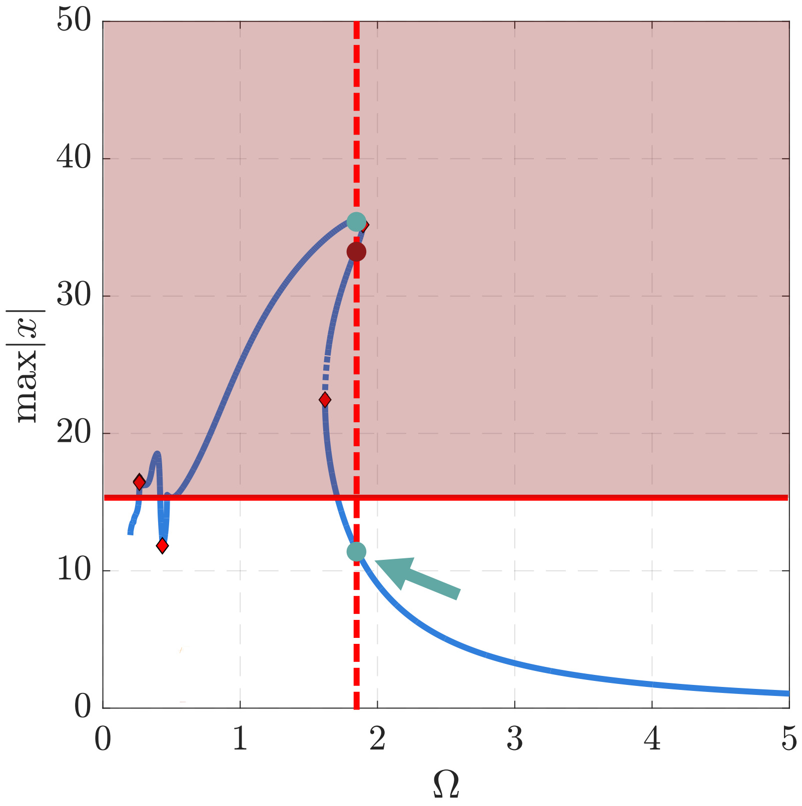


This means that if we excite the system under consideration with a dimensionless forcing of the type $\mathrm{F}\cos(\Omega\tau)$

where $\mathrm{F} = 20$ and $\Omega = 1.859$ the system will exhibit:

- 3 periodic orbits, 2 **stable** and 1 **unstable**;

Numerical Applications



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where $F = 20$ and $\Omega = 1.859$ the system will exhibit:

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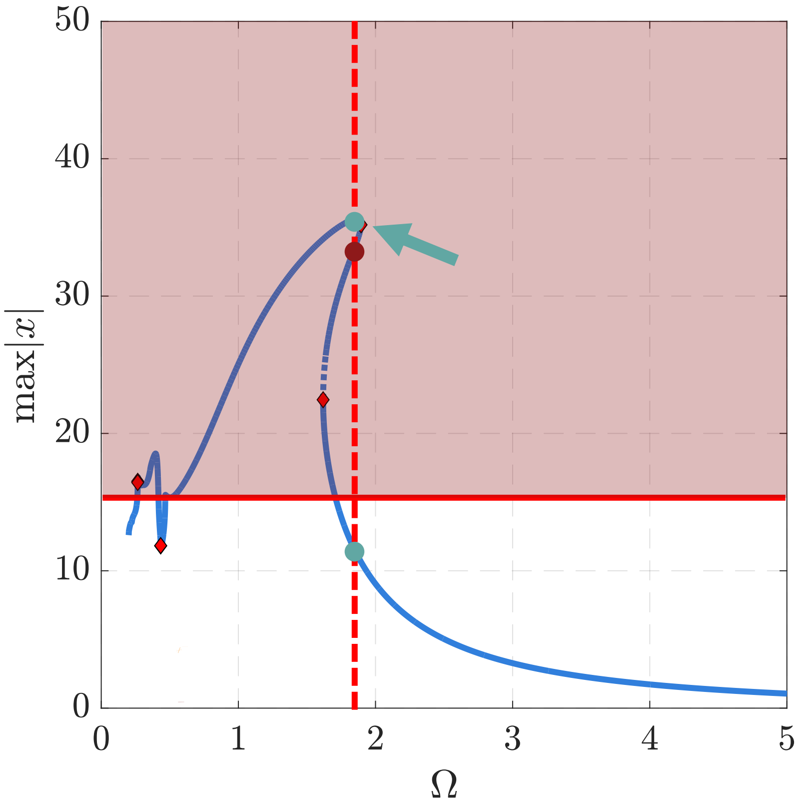
- Among the two stable periodic orbits, one has a smaller maximum displacement compared to the displacement used to calibrate the model parameters

Numerical Applications



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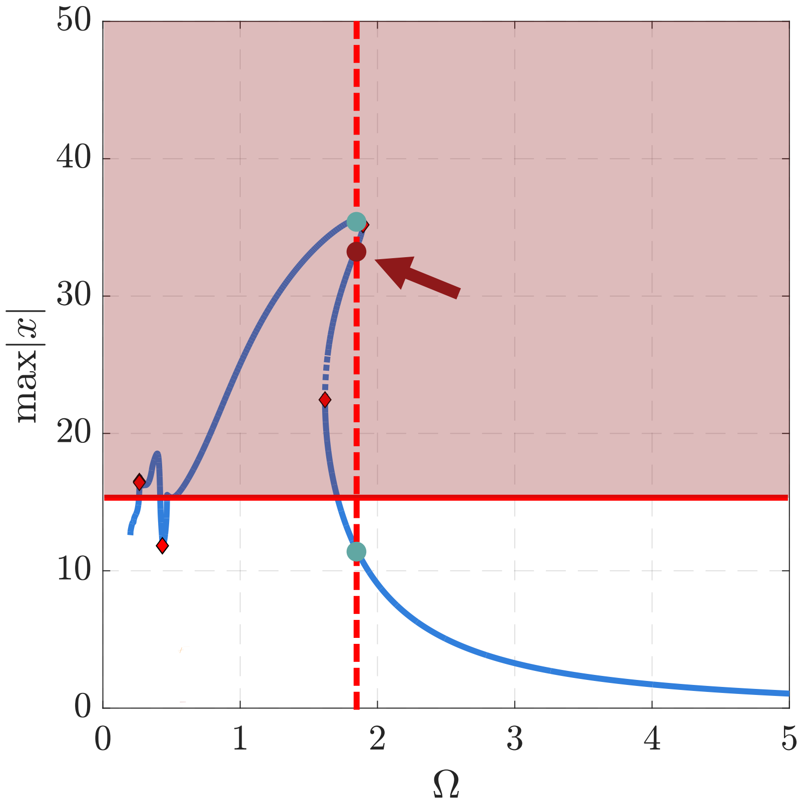
Numerical Applications



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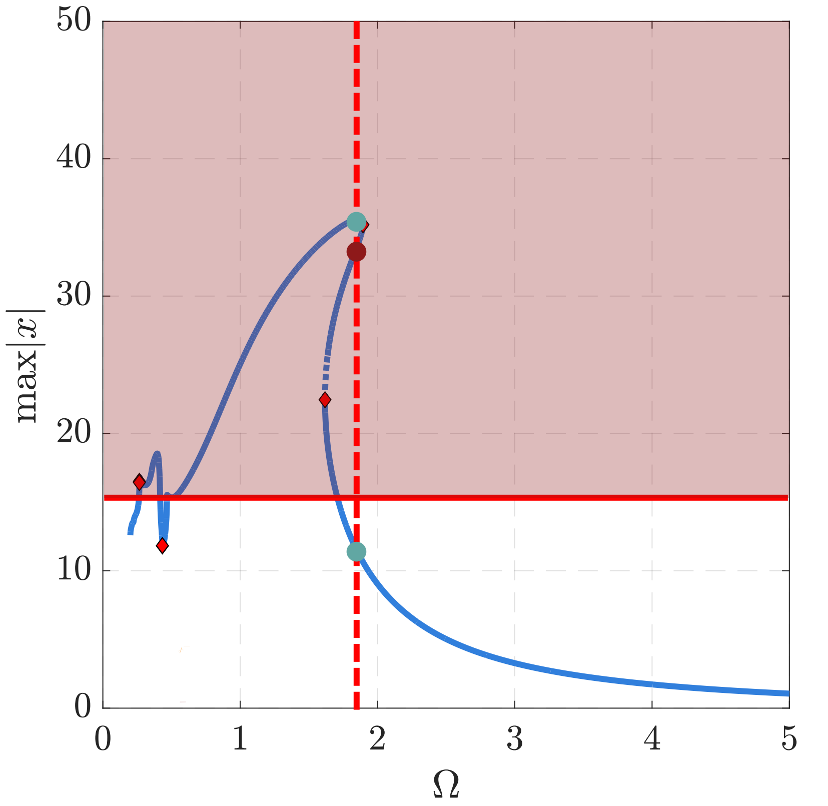
- The unstable periodic orbit has a larger displacement than the one used for calibrating the model parameters.

Numerical Applications



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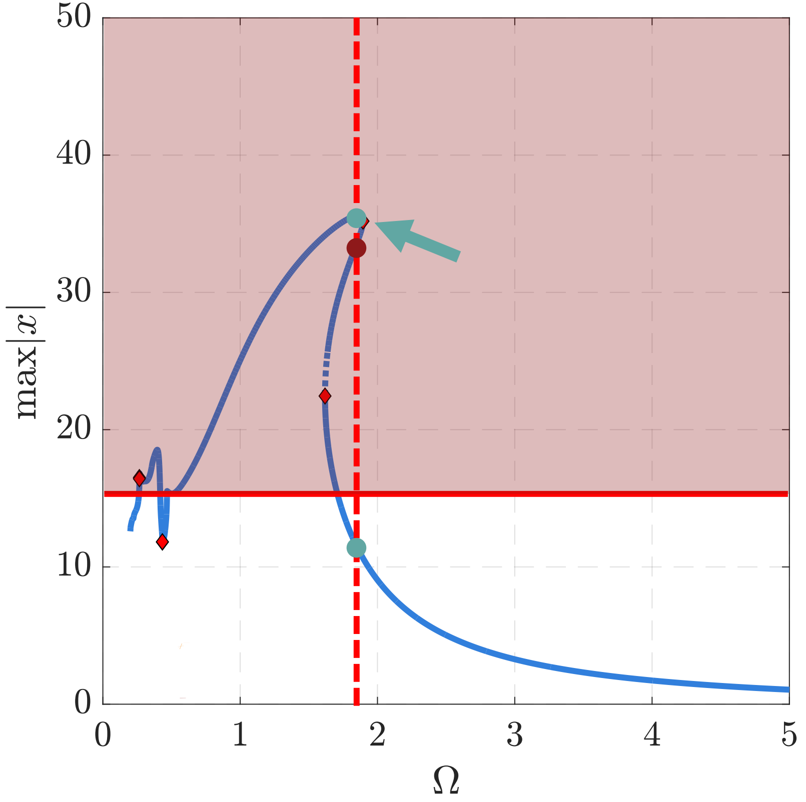
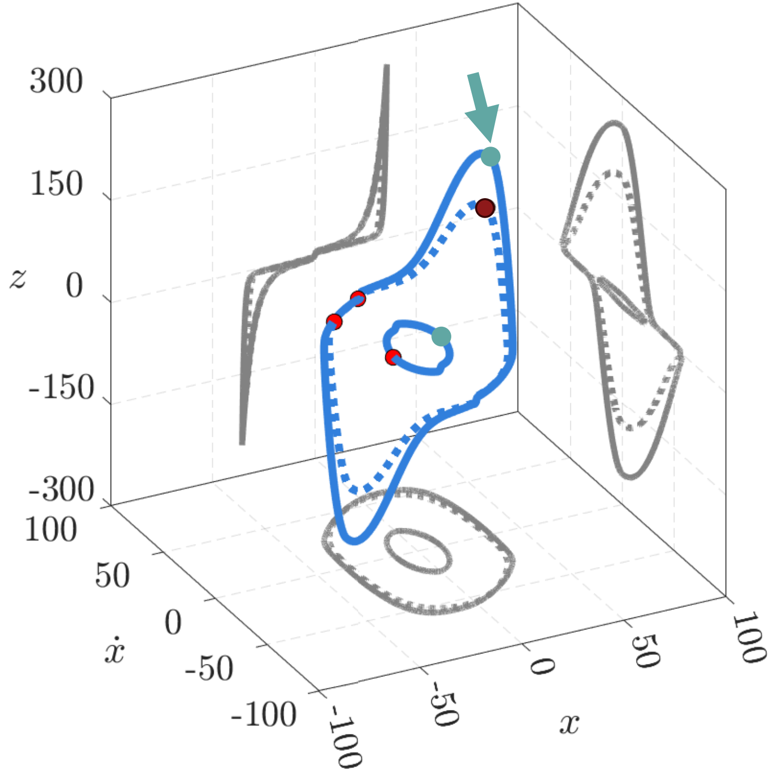
The conclusion we can draw is that the behavior of the specific system studied is dependent on the initial conditions for $F = 20$ e $\Omega = 1.859$.
In particular, there may be specific initial conditions for which the result obtained from a NLTH is not acceptable.

Numerical Applications





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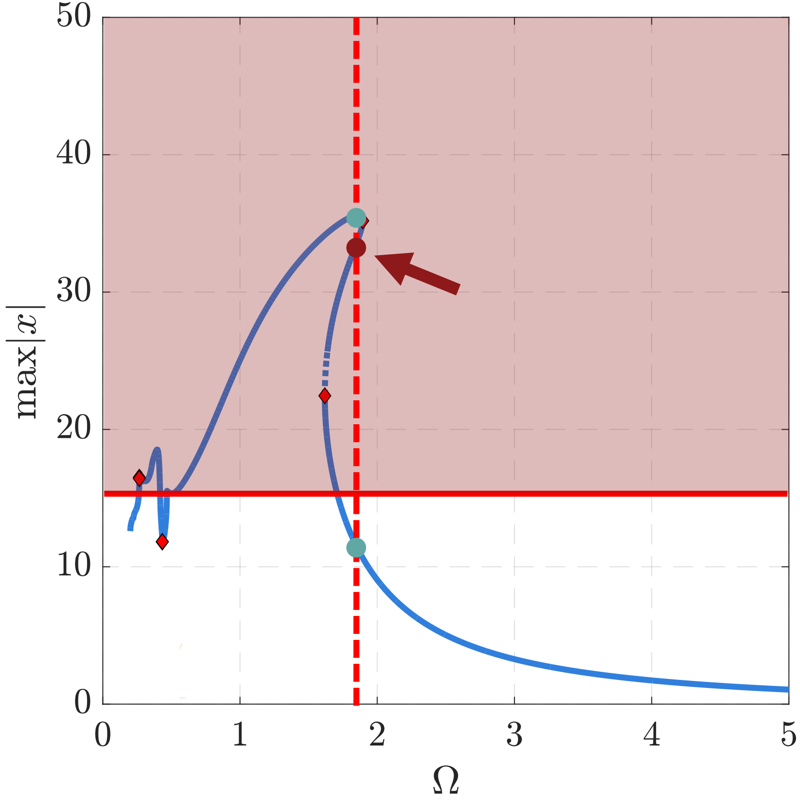

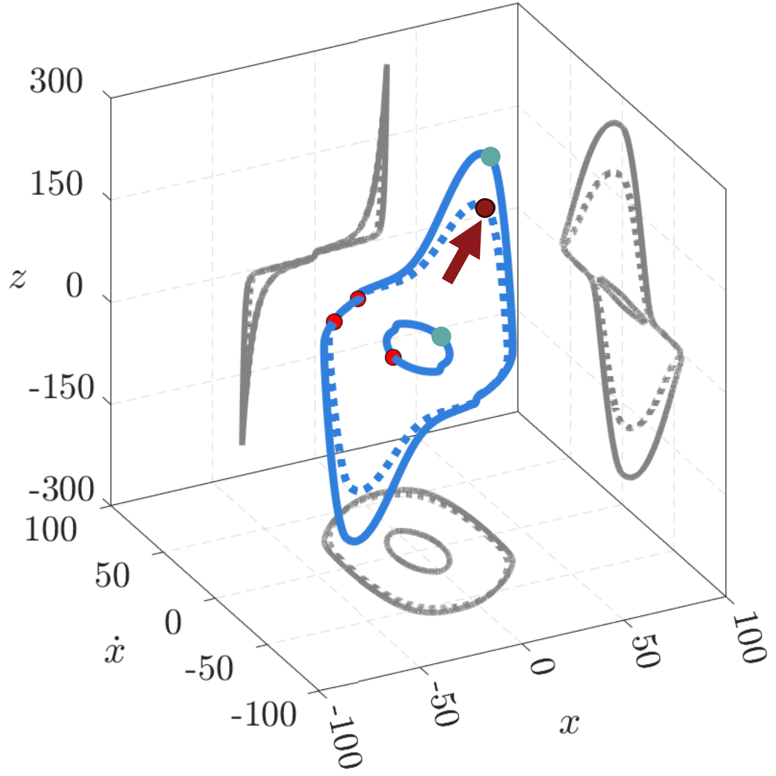

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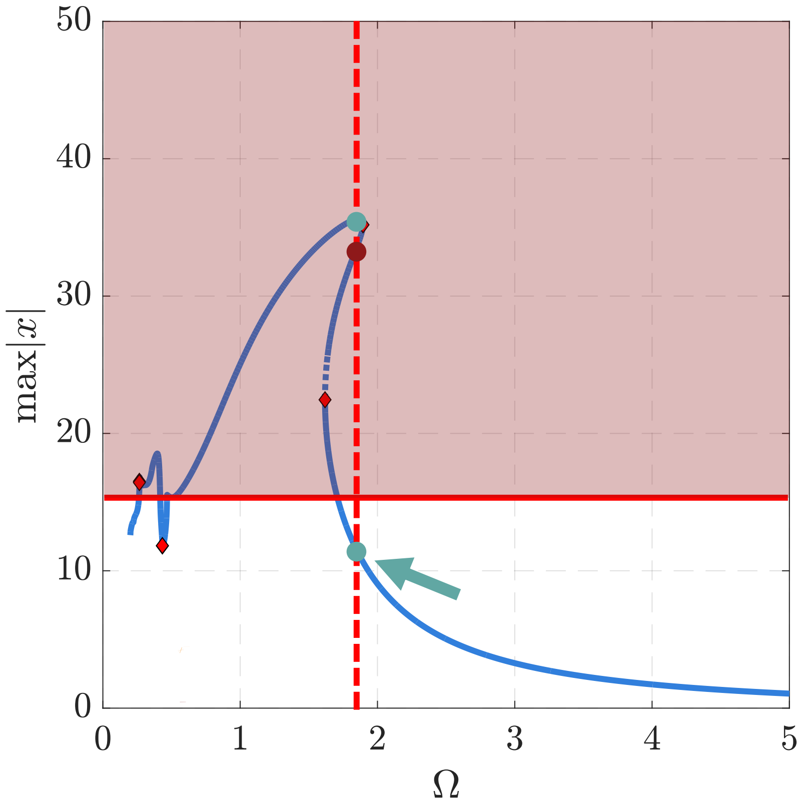

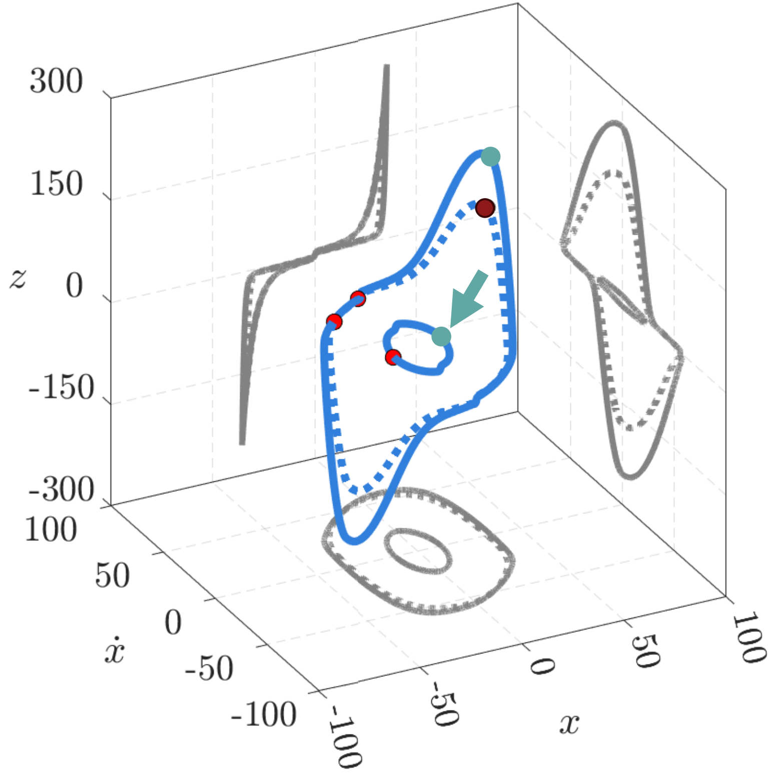

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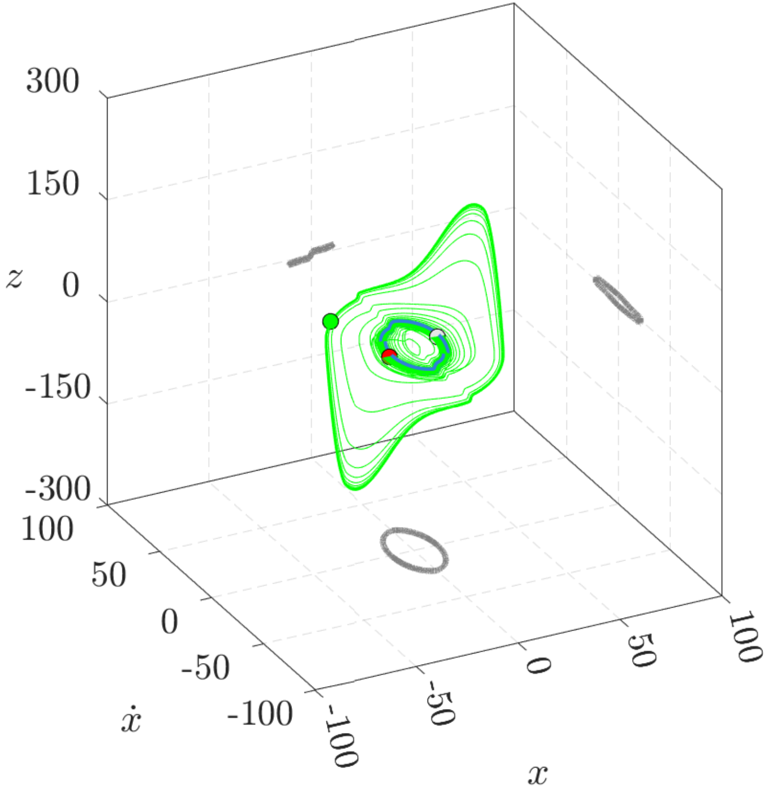

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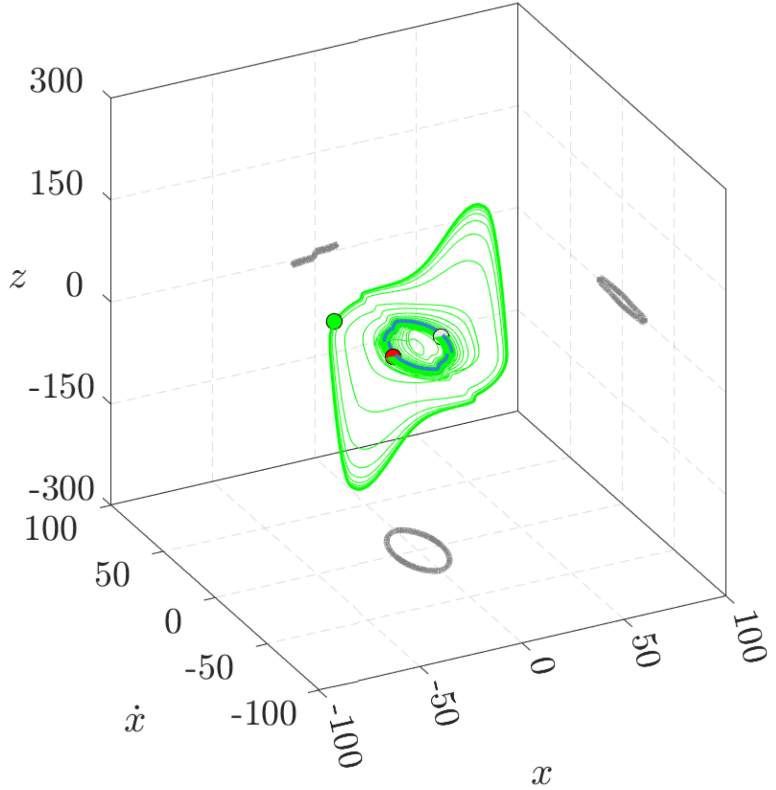

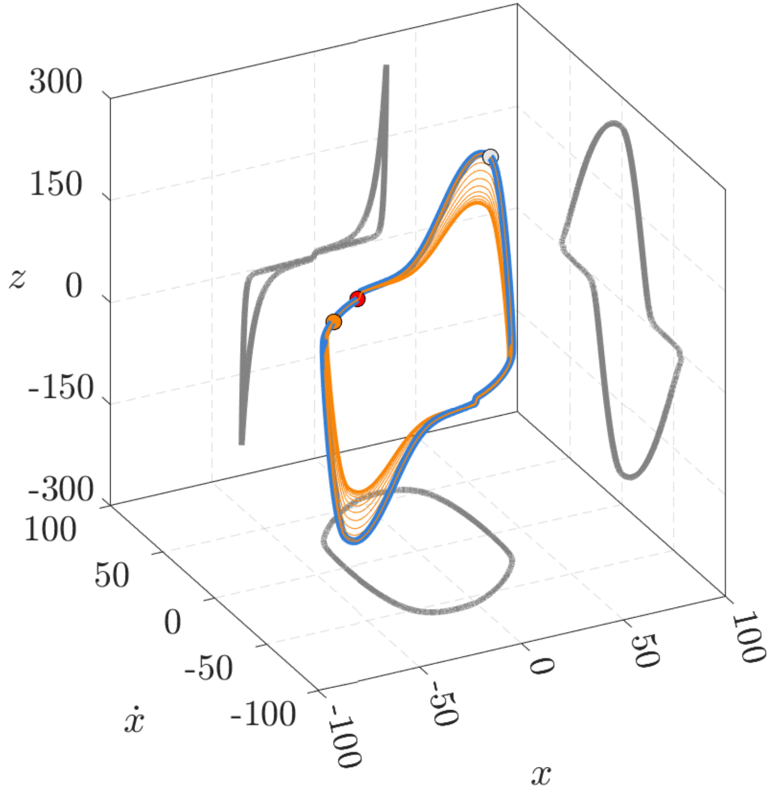

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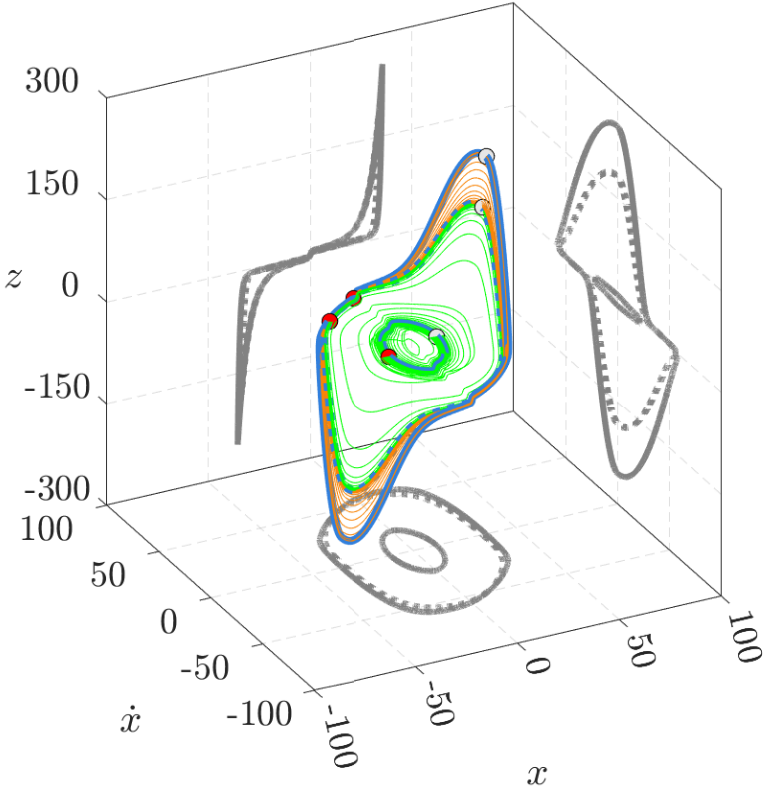

National Technical University of Athens

SCHOOL OF APPLIED MATHEMATICAL AND PHYSICAL SCIENCES

Thanks for your attention!

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Department of Structures for Engineering and Architecture