Determination of SIF in a cracked plane orthotropic strip by the Wiener–Hopf technique

H.G. GEORGIADIS and G.A. PAPADOPoulos
133–35 G. Papandreou Str., 16231 Athens, Greece; 2Division of Mechanics, Department of Engineering Sciences, The National Technical University of Athens, 15773 Athens, Greece

Received 21 October 1986; accepted 26 December 1986

Abstract. The stress intensity factor for a long cracked strip was determined within the context of the linear orthotropic elasticity. The body had the form of an infinite strip containing a semi-infinite crack at the middle distance of the strip faces. Fourier transforms in combination with the Wiener–Hopf technique were employed to evaluate asymptotically the cleavage stress and its intensity at the crack tip.

1. Introduction

In the realm of linear elastic fracture mechanics (LEFM), the main interest is in the elastic stress intensity factor at the tip of cracks or sharp notches. As is well-known the determination of this factor is indispensable for the application of the Griffith–Irwin fracture concepts [1, 2]. In the early years of development of LEFM, researchers confined themselves to finding solutions for crack problems in bodies of infinite dimensions. In recent years, however, there is an increasing interest in analytical solutions in cracked finite bodies. In addition, more general constitutive relations are considered for the mechanical response of the cracked solids involving anisotropy, inhomogeneity and non-linearity. As a consequence, the mathematical crack problems become more and more difficult and the techniques for their solution more and more sophisticated.

In this paper we have considered a cracked long strip made by an orthotropic elastic material involving four independent elastic constants. The strip contains a semi-infinite crack parallel to and equally spaced from the strip faces. The corresponding isotropic case of the foregoing problem was treated by Knauss [3] (see also discussion and correction by Rice [19]) whereas Nilsson [4] considered the isotropic steady-state elastodynamic case. Recently, Sih and Chen in a treatise on cracked composite bodies [5] considered many crack problems of a similar fashion, but the present case was not included. Infinite orthotropic strips with internal cracks were also considered by Konishi and Atsumi [6] and Satapathy and Parhi [7]. It is also worthwhile to mention some other analytical works concerning the double cantilever beam configuration (which involves a cracked strip loaded by concentrated forces near the crack faces) carried out by Kanninen [8, 9] and Fichter [10].

2. Basic preliminaries

The material we considered was orthotropic with two mutually orthogonal axes of elastic symmetry in the plane. With respect to the principal material-axes of an orthotropic plate,
the elastic constitutive expression relating the in-plane stresses and strains is [11]

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix} =
\begin{bmatrix}
c_{11} & c_{12} & 0 \\
c_{12} & c_{22} & 0 \\
0 & 0 & c_{66}
\end{bmatrix}
\begin{bmatrix}
e_x \\
e_y \\
\gamma_{xy}
\end{bmatrix}.
\]

(1)

The coefficients of the stiffness matrix are expressible in terms of the engineering elastic constants by

\[
c_{11} = E_1/(1 - \nu_1\nu_2), \quad c_{12} = \nu_2 E_1/(1 - \nu_1\nu_2)
\]

\[
c_{22} = E_2/(1 - \nu_1\nu_2), \quad c_{66} = G.
\]

(2)

The subscripts 1 and 2 refer to the principal directions of material symmetry which coincide here with the \(x\) and \(y\) reference axes. In the above relations, only four elastic constants are independent, the fifth given by

\[
v_2 E_1 = \nu_1 E_2.
\]

(3)

Then, the equations of equilibrium yield the following expressions

\[
c_{11} \frac{\partial^2 u_x}{\partial x^2} + c_{66} \frac{\partial^2 u_x}{\partial y^2} + (c_{12} + c_{66}) \frac{\partial^2 u_x}{\partial x \partial y} = 0
\]

(4.1)

\[
c_{66} \frac{\partial^2 u_y}{\partial x^2} + c_{22} \frac{\partial^2 u_y}{\partial y^2} + (c_{12} + c_{66}) \frac{\partial^2 u_y}{\partial x \partial y} = 0
\]

(4.2)

which determine the two unknowns \(u_x\) and \(u_y\).

Consider now a displacement-potential formulation, analogous to that utilized frequently in elastodynamics

\[
u_x = \frac{\partial}{\partial x} (\phi + \psi), \quad u_y = \frac{\partial}{\partial y} (a\phi + b\psi), \quad u_z = 0
\]

(5)

where \(\phi\) and \(\psi\) are the displacement potentials and the new constants are given by

\[
a = \frac{c_{11}\beta_1 - c_{66}}{c_{12} + c_{66}} = \frac{(c_{12} + c_{66})\beta_1}{c_{22} - c_{66}\beta_1}, \quad b = \frac{c_{11}\beta_2 - c_{66}}{c_{12} + c_{66}} = \frac{(c_{12} + c_{66})\beta_2}{c_{22} - c_{66}\beta_2}.
\]

(6)

In (6) \(\beta_1\) and \(\beta_2\) are the roots of the characteristic equation

\[
c_{11}c_{66}\beta^2 + (c_{12} + 2c_{12}c_{66} - c_{11}c_{22})\beta + c_{22}c_{66} = 0.
\]

(7)
Substituting relations (5) into (4) leads to the following Laplace type equations satisfied by the displacement potentials

$$\frac{\partial^2 \phi}{\partial x^2} + \beta_1 \frac{\partial^2 \phi}{\partial y^2} = 0, \quad \frac{\partial^2 \psi}{\partial x^2} + \beta_2 \frac{\partial^2 \psi}{\partial y^2} = 0. \quad (8)$$

Finally, stresses are given in terms of the displacement potentials by

$$\sigma_x = \frac{(1 + a)c_{66}}{\beta_1} \frac{\partial^2 \phi}{\partial y^2} + \frac{(1 + b)c_{66}}{\beta_2} \frac{\partial^2 \psi}{\partial y^2}, \quad (9.1)$$

$$\sigma_y = -(1 + a)c_{66} \frac{\partial^2 \phi}{\partial x^2} - (1 + b)c_{66} \frac{\partial^2 \psi}{\partial x^2}, \quad (9.2)$$

$$\tau_{xy} = c_{66} \left[ (1 + a) \frac{\partial^2 \phi}{\partial x \partial y} + (1 + b) \frac{\partial^2 \psi}{\partial x \partial y} \right]. \quad (9.3)$$

Solution therefore of boundary value problems for orthotropic bodies in plane strain may be accomplished by solving (8) under the imposed boundary conditions which involve (5) and (9). As will be seen later, the above displacement potential formulation is more advantageous than the Airy stress function formulation utilized to solve other boundary value problems involving strip-like orthotropic bodies [12].

3. Analysis

As depicted in Fig. 1, we consider an elastic orthotropic body in the form of an infinitely long strip of height $2h$. The layer contains a semi-infinite crack which opens in plane extension under the action of constant displacements applied to the strip faces. The boundary
conditions of the problem in the upper half-plane are written as

\[
\begin{align*}
\tau_{yy}(x, h) &= 0 \quad \text{for } -\infty < x < \infty \quad (10.2) \\
\tau_{xx}(x, 0) &= 0 \quad \text{for } -\infty < x < \infty \quad (10.3) \\
\sigma_y(x, 0) &= 0 \quad \text{for } -\infty < x < 0 \quad (10.4) \\
u_y(x, 0) &= 0 \quad \text{for } 0 < x < \infty. \quad (10.5)
\end{align*}
\]

Since it is known from similar cases [4, 13] that this form of boundary conditions is not convenient for applying the Wiener-Hopf technique, we consider the following auxiliary problem

\[
\begin{align*}
u_y(x, h) &= 0 \quad \text{for } -\infty < x < \infty \quad (11.1) \\
\tau_{xy}(x, h) &= 0 \quad \text{for } -\infty < x < \infty \quad (11.2) \\
\tau_{xx}(x, 0) &= 0 \quad \text{for } -\infty < x < \infty \quad (11.3) \\
\sigma_y(x, 0) &= \sigma_0 \quad \text{for } -\infty < x < 0 \quad (11.4) \\
u_y(x, 0) &= 0 \quad \text{for } 0 < x < \infty. \quad (11.5)
\end{align*}
\]

Then, by a trivial superposition we arrive again at the original problem. The proper value of \(\sigma_0\) in the foregoing boundary conditions and for plane strain results by (1) as

\[
\sigma_0 = -c_{22}u_0/h. \quad (12)
\]

Now we introduce two as yet unknown functions the determination of which completes the solution of problem (11)

\[
\begin{align*}
\sigma_y(x, 0) &= m(x) \quad \text{for } 0 < x < \infty \quad (13.1) \\
u_y(x, 0) &= n(x) \quad \text{for } -\infty < x < 0. \quad (13.2)
\end{align*}
\]

The existence of infinite and semi-infinite domains in the boundary conditions in addition to (8) implies the suitability of the Wiener-Hopf method. This method, as noted in [14], was originated by Carleman. The treatise of Noble [15] is the standard reference for this technique and a brief account is also given in Carrier et al. [14]. To apply this, consider the following exponential Fourier transform pair [14, 16]

\[
f^*(\omega, y) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} f(x, y) e^{i\omega x} \ dx \quad (14.1)
\]
\[ f(x, y) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} f^*(\omega, y) e^{-i\omega x} d\omega \] (14.2)

where \( \omega = \sigma + \tau \) is the complex variable in the Fourier-transform plane.

Applying the operation (14) to (8) gives

\[ -\omega^2 \phi^*(\omega, y) + \beta_1 \frac{\partial^2}{\partial y^2} \phi^*(\omega, y) = 0 \] (15.1)

\[ -\omega^2 \psi^*(\omega, y) + \beta_2 \frac{\partial^2}{\partial y^2} \psi^*(\omega, y) = 0. \] (15.2)

Obviously, Eqns. (15) can be regarded as ordinary differential equations for \( \phi^*(\omega, y) \) and \( \psi^*(\omega, y) \) considered as functions of the \( y \)-variable. General solutions of (15) are of the form

\[ \phi^*(\omega, y) = A(\omega) e^{\gamma_1 y} + B(\omega) e^{-\gamma_1 y} \] (16.1)

\[ \psi^*(\omega, y) = C(\omega) e^{\gamma_2 y} + D(\omega) e^{-\gamma_2 y} \] (16.2)

where \( \gamma_j = \beta_j^{-1/2} \) (\( j = 1, 2 \)).

On the other hand, stresses and displacements involved in the boundary conditions take the following form in the Fourier-transform plane

\[ u^*(\omega, y) = \frac{\partial}{\partial y} [a\phi^*(\omega, y) + b\psi^*(\omega, y)] \] (17.1)

\[ \sigma_y^*(\omega, y) = c_{66} \omega^2 [(1 + a)\phi^*(\omega, y) + (1 + b)\psi^*(\omega, y)] \] (17.2)

\[ \tau_{xy}^*(\omega, y) = -ic_{66} \omega \frac{\partial}{\partial y} [(1 + a)\phi^*(\omega, y) + (1 + b)\psi^*(\omega, y)]. \] (17.3)

Consider now the transforms of the functions of interest

\[ m^*_+(\omega) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} m(x) e^{i\omega x} dx \] (18.1)

\[ n^*_-(\omega) = (2\pi)^{-1/2} \int_{-\infty}^{0} n(x) e^{i\omega x} dx. \] (18.2)

From the physics of the problem it is reasonable to assume that the functions \( m(x) \) and \( n(x) \) are exponentially bounded at infinity and this ensures the existence of their Fourier transform. In particular, we may state that [15]

\[ |m(x)| < M \exp (\tau_- x), \quad x \to +\infty \to m^*_+(\omega) \text{ is analytic in } \tau > \tau_- = \tau_1 \]

\[ |n(x)| < N \exp (\tau_+ x), \quad x \to -\infty \to n^*_-(\omega) \text{ is analytic in } \tau < \tau_+ = \tau_2 \]
where the (+) and (−) subscripts denote that the functions are analytic above or below a certain line in the complex ω-plane.

Then, application of the Fourier transform and consideration of the boundary conditions (11), (13) in conjunction with (16)–(18) results in an algebraic system of five equations with the six unknown functions $A(ω), B(ω), C(ω), D(ω), m^*(ω)$ and $n^*(ω)$. After some lengthy algebra the first four unknowns were eliminated and the system was reduced to the following Wiener−Hopf equation

$$m^*(ω) = \frac{c_6(1 + a)(1 + b)ω}{(b - a)γ_1γ_2} [γ_2 \coth (γ_1ωh) - γ_1 \coth (γ_2ωh)] n^*(ω) - \frac{σ_0}{i(2π)^{1/2}ω}$$

(19)

where the kernel is

$$K(ω) = ω[γ_2 \coth (γ_1ωh) - γ_1 \coth (γ_2ωh)]$$

(20)

which must be factored into the product form

$$K(ω) = K_+(ω)K_-(ω).$$

(21)

As a consequence of the above, the Wiener-Hopf relationship (19) may take the form

$$\frac{m^*(ω)}{K_+(ω)} + \frac{σ_0}{i(2π)^{1/2}ω} \frac{1}{K_+(ω)} = \frac{c_6(1 + a)(1 + b)}{(b - a)γ_1γ_2} K_-(ω)n^*(ω)$$

(22)

which is basic for the application of the method. Due to the fact that the kernel in our problem may be factored in closed form with much difficulty, we confine ourselves to find asymptotic results near the origin (0, 0). However, this fully covers the purposes of LEFM since we can estimate thus the cleavage stress ahead of the crack tip and then easily its intensity. In what follows, we adopt a procedure introduced by Nilsson [4] which was also utilized in some subsequent works [13, 17, 18].

We take

$$E(ω) = \frac{σ_0}{i(2π)^{1/2}ω} \frac{1}{K_+(ω)} = E_+(ω) + E_-(ω)$$

(23)

with

$$E_+(ω) = \frac{σ_0}{i(2π)^{1/2}ω} \left[ \frac{1}{K_+(ω)} - \frac{1}{K_+(ω)} \right]$$

(24.1)

$$E_-(ω) = \frac{σ_0}{i(2π)^{1/2}ω} \frac{1}{K_+(0)}.$$
In view of (23) and (24), relation (22) becomes
\[
\frac{m^*(\omega)}{K_+^{\mkern-1mu}(\omega)} + E_+^{\mkern-1mu}(\omega) = \frac{c_{66}(1 + a)(1 + b)}{(b - a)\gamma_1\gamma_2} K_-(\omega)n^*(\omega) - E_-(\omega) \equiv J(\omega). \tag{25}
\]

The following considerations are now in order: The first member in (25) is analytic in the upper half plane \( \text{Im} \, \omega = \tau > \max (\tau_1, \tau_2) < 0 \) and the second member in the lower half plane \( \text{Im} \, \omega = \tau < 0 \). Therefore, the regions of analyticity overlap and invoking analytic continuation it is concluded that \( J(\omega) \) is analytic and single-valued in the whole \( \omega \)-plane. Furthermore, by using the arguments in [15] (p. 37, 38) it is found that \( J(\omega) = 0 \). Then, the functions of interest readily resulted by (23)–(25) as
\[
m^*(\omega) = -\frac{\sigma_0 K_+(\omega)}{i(2\pi)^{1/2} \omega} \left[ \frac{1}{K_+(\omega)} - \frac{1}{K_+(0)} \right], \tag{26.1}
\]
\[
n^*(\omega) = \frac{\sigma_0 c_{66}(1 + a)(1 + b)}{i(2\pi)^{1/2} \omega (b - a)\gamma_1\gamma_2} \frac{1}{K_-(\omega)K_+(0)}. \tag{26.2}
\]

Thus, we have to determine the asymptotic values of the kernel \( K(\omega) \) for \( \omega \to \infty \) and \( \omega \to 0 \). It was found that
\[
\lim_{\omega \to \infty} \frac{K(\omega)}{\omega} = \gamma_2 - \gamma_1 \tag{27}
\]
\[
\lim_{\omega \to 0} K(\omega) = \frac{\gamma_2^2 - \gamma_1^2}{\gamma_1\gamma_2 h}. \tag{28}
\]

Furthermore, by noting that \( K(\omega) \) is an even function and using a table of inverse Fourier transforms [16], we take
\[
\lim_{\omega \to \infty} m^*(\omega) = -\frac{\sigma_0}{i(2\pi)^{1/2} \omega} + \frac{\sigma_0}{i(2\pi)^{1/2} \omega^{1/2} K_+(0)} \lim_{\omega \to \infty} \frac{K_+(\omega)}{\omega^{1/2}} \to
\]
\[
\lim_{x \to 0^+} m(x) = -\frac{\sigma_0}{\pi^{1/2}} \left( \frac{\gamma_1\gamma_2 h}{\gamma_1 + \gamma_2} \right)^{1/2} x^{-1/2} \tag{29}
\]
where we have ignored the constant term, and
\[
\lim_{\omega \to \infty} n^*(\omega) = \frac{\sigma_0 c_{66}(1 + a)(1 + b)}{i(2\pi)^{1/2} \omega^{3/2} (b - a)\gamma_1\gamma_2} \frac{1}{K_+(0)} \lim_{\omega \to \infty} \frac{\omega^{1/2}}{K_-(\omega)} \to
\]
\[
\lim_{x \to 0^-} n(x) = \frac{\sigma_0 c_{66}(1 + a)(1 + b)}{i\pi^{1/2} (b - a)\gamma_1\gamma_2 (\gamma_2 - \gamma_1)^{1/2}} \left( \frac{\gamma_1\gamma_2 h}{\gamma_2 - \gamma_1} \right)^{1/2} x^{1/2}. \tag{30}
\]

Relations (29) and (30) are the asymptotic expressions of the \( \sigma_s(x, 0) \) – stress and \( u_s(x, 0) \) – displacement, i.e., of the cleavage stress and crack shape, respectively, near the crack tip.
Then the stress intensity factor at the crack tip can be obtained by the relation

\[ K_1 = \lim_{z \to 0} [(2\pi x)^{1/2} \sigma(x, 0)]. \]  

(31)

In view of (12), (29) and (31) it is found that

\[ K_1 = \left( \frac{2\gamma_1 \gamma_2}{\gamma_1 + \gamma_2} \right)^{1/2} \frac{c_{22} h_0}{h^{1/2}} \]  

(32)

which for the isotropic case is reduced to the value estimated by Rice [19].

4. Conclusions

By utilizing integral transforms and the Wiener–Hopf technique for the asymptotic solution of the governing differential equations we easily obtained the stress ahead of the crack tip and the crack shape in a cracked long orthotropic strip. The SIF dependence upon the orthotropic elastic constants and the strip depth were also found.

References