THE STATIONARY VALUE OF THE THIRD STRESS INVARIANT AS A LOCAL FRACTURE PARAMETER (DET.-CRITERION)

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Abstract—In this paper the stationary value of the third stress invariant as a local fracture parameter is studied. The third stress invariant $\text{Det}(\sigma_{ij})$ is calculated along a circle around the crack-tip. This circle, which defines the core-region around the crack-tip, is the initial curve of the caustics. This distribution of the $\text{Det}(\sigma_{ij})$ presents a positive maximum. The crack propagates in the direction of the maximum value of the $\text{Det}(\sigma_{ij})$ and the fracture will initiate when the $\text{Det}(\sigma_{ij})$ on the core-region boundary reaches a critical maximum value which is a material property. This condition of initiation of the crack is proposed as Det.-criterion of fracture.

INTRODUCTION

VARIOUS criteria have been introduced for the determination the conditions of initiation and the prediction of the propagation direction of a crack submitted to a combined mode I and mode II in-plane deformations. First of all, Griffith[1] introduced a criterion based on physical considerations to determine the conditions to initiate the propagation of a crack. Erdogan and Sih[2] developed the maximum tangential stress criterion. According to this criterion the crack will propagate along the direction of the maximum tangential stress. The maximum energy release rate criterion, $G$-criterion,[3, 4] represents a generalization of Griffith’s original energy-release-rate concept[1]. Sih[5] developed the minimum elastic strain-energy density criterion (S-criterion) which is based directly on the total strain-energy density, that is, the sum of its distortional and dilatational components. Theocaris and Papadopoulos[6] proposed the two-term approximation $S_2$-criterion which is a modification of the usual S-criterion, with a higher approximation which considers two terms in the series expansion of the stress function[7], instead of one term (the singular one), as anticipated by Sih. Modifications on the S-criterion have been done by Theocaris and Andrianopoulos[8] and Wang[9]. Recently, Theocaris and Andrianopoulos[10, 11] proposed a new criterion referred to as the $T$-criterion. According to $T$-criterion the crack propagates along a direction defined by a maximum of the total energy density, which is also a maximum for the dilatational strain-energy density when this distribution is evaluated along a locus of constant distortional strain-energy density, as it is the initial Mises elastic–plastic boundary. Also, the fracture will initiate when the dilatational strain-energy density on the elastic–plastic boundary reaches a critical maximum value which is a material property. Yehia[12] proposed a modification on $T$-criterion that the fracture will initiate when the distance from the crack-tip to the elastic–plastic boundary in the direction of crack propagation reaches a maximum value.

In this paper a simple criterion of fracture based on the third stress invariant will be developed.

THEORETICAL CONSIDERATION

The elastic strain energy $dW$ stored in a parallel-piped of volume $dV$ dominating at the strained plate is expressed by [13]:

$$\frac{dW}{dV} = \frac{1 + \nu}{4E} \left[ \kappa_{1,2}(\sigma_x + \sigma_y)^2 + (\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2 \right]$$  

(1)
where $E$ is the modulus of elasticity of the material, $\nu$ is the Poisson ratio and $\sigma_x$, $\sigma_y$ and $\tau_{xy}$ are the components of the stress tensor $\sigma_{ij}$ and $\kappa_{1,2}$ take the values:

$$\kappa_1 = \frac{1 - \nu}{1 + \nu}, \text{ for plane stress}$$

$$\kappa_2 = 1 - 2\nu, \text{ for plane strain.}$$

The total elastic strain-energy density (1) is divided into two components, the dilatational strain-energy density $T_V$ and the distortional strain-energy density $T_D$. For plane-stress and plane-strain problems the components $T_V$ and $T_D$ are given by [14, 15]:

$$T_V = \frac{1 - 2\nu}{6E} (\sigma_x + \sigma_y)^2$$  \hspace{1cm} (3)

$$T_D = \frac{1 + \nu}{3E} \left[ (\sigma_x + \sigma_y)^2 - 3(\sigma_x\sigma_y - \tau_{xy}^2) \right]$$  \hspace{1cm} (4)

for plane-stress, and

$$T_V = \frac{(1 - 2\nu)(1 + \nu)^2}{6E} (\sigma_x + \sigma_y)^2$$

$$T_D = \frac{1 + \nu}{3E} \left[ (\nu^2 - \nu + 1)(\sigma_x + \sigma_y)^2 - 3(\sigma_x\sigma_y - \tau_{xy}^2) \right]$$

for plane-strain.

From the relations (4) and (6) the elastic–plastic boundary can be obtained by setting the distortional strain-energy density $T_D$ equal to the maximum constant value $T_{D,0}$ which is given by the relation:

$$T_D = T_{D,0} = \frac{1 + \nu}{3E} \sigma_0^2$$

where $\sigma_0$ is the yield stress of the material.

Substituting relations (3) and (5) into relations (4) and (6), respectively, we obtain:

$$T_D = \frac{2(1 + \nu)}{1 - 2\nu} T_V - \frac{1 + \nu}{E} (\sigma_x\sigma_y - \tau_{xy}^2), \text{ for plane-stress}$$

$$T_D = \frac{2(\nu^2 - \nu + 1)}{(1 - 2\nu)(1 + \nu)} T_V - \frac{1 + \nu}{E} (\sigma_x\sigma_y - \tau_{xy}^2), \text{ for plane-strain}$$

or respectively,

$$T_V = \frac{1 - 2\nu}{2(1 + \nu)} T_D + \frac{1 - 2\nu}{2E} \text{Det}(\sigma_{ij})$$

$$T_V = \frac{(1 - 2\nu)(1 + \nu)}{2(\nu^2 - \nu + 1)} T_D + \frac{(1 - 2\nu)(1 + \nu)^2}{2(\nu^2 - \nu + 1)E} \text{Det}(\sigma_{ij})$$
where

$$\text{Det}.(\sigma_{ij}) = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{vmatrix}$$  \hspace{1cm} (12)$$

is the third stress invariant.

Relations (10) and (11) show that the dilatational strain-energy density $T_V$ depends on the distortional strain-energy density $T_D$ and the third invariant of stress tensor, $\text{Det}.(\sigma_{ij})$. So, for a constant value of $T_D$, the $T_V$ reaches a maximum value. According to the $T$-criterion [10, 11, 16]: (i) The maximum value of the dilatational component of the elastic strain-energy density $T_V$ reaches a critical value for $T_V$, say $T_{V,0}$, which is a material property, (ii) The distortional strain-energy density $T_D$ around the crack tip must satisfy the condition $T_D = T_{D,0} = \text{const.}$ and (iii) The dilatational component $T_V$ around the crack tip, evaluated along the respective yield locus where $T_D - T_{D,0}$ should present a maximum in the front region of the crack. A crack starts to propagate if this maximum overpasses a critical values $T_{V,0}$ for this component of energy density.

From the above it is concluded that; for $T_D = T_{D,0}$, the maximum value of $T_V$ and its position strongly depends on the term $\text{Det}.(\sigma_{ij})$ of relations (10) and (11). Therefore, the $\text{Det}.(\sigma_{ij})$ arranges the two components $T_V$ and $T_D$ of the total elastic strain-energy density around the crack tip, so that, when $T_D = T_{D,0} = \text{const.}$ the maximum value of the $\text{Det}.(\sigma_{ij})$ gives $T_V$ the maximum value $T_{V,\text{max}}$ and subsequently crack propagation will occur. In polar coordinate system $(r, \theta)$ centered at crack-tip (Fig. 1), this condition may be expressed by:

$$\left. \frac{\partial \text{Det}.(\sigma_{ij})}{\partial \theta} \right|_{\theta = \theta_0} = 0, \quad \left. \frac{\partial^2 \text{Det}.(\sigma_{ij})}{\partial \theta^2} \right|_{\theta = \theta_0} < 0. \hspace{1cm} (13)$$

Relation (13), calculated along a circle around the crack-tip, with relations (10) and (11) defines the following condition:

$$T_{V,\text{max}} = T_V|_{\theta = \theta_0} \equiv T_{V,0} = \text{const.} \hspace{1cm} (14)$$

Relation (13) describes a new criterion of fracture which we call as $\text{Det}.$-criterion: The $\text{Det}.$-criterion postulates that the crack propagates along the direction defined by a maximum of the third stress invariant when this distribution is evaluated along a circle, initial curve of caustics, around the crack-tip. On the contrary, the $T$-criterion is based on the distribution of

Fig. 1. Geometry of a cracked plate, submitted at infinity to a tension, with internal slant-crack.
the dilatational strain-energy density $T_V$ along a locus of constant distortional strain-energy density $T_D$, as it is the initial Mises elastic-plastic boundary. So, the direction $\theta_0$ of crack initiation is defined by the relation (13) and the critical stress of crack initiation is calculated by:

$$\text{Det}(\sigma_{ij}) = \text{Det}(\sigma_{ij})_{cr}. \quad (15)$$

**APPLICATION OF DET.-CRITERION**

For a thin elastic and isotropic plate, under conditions of generalized plane stress, containing a slant internal crack of length $2\alpha$ and obliqueness $\beta$ (Fig. 1), which is submitted at infinity to a tension $\sigma_\infty$, the stress field in the vicinity of the crack tip is given by $[7, 17, 18]$:

$$\sigma_x = \frac{K_1}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2}\right) - \frac{K_{II}}{\sqrt{2\pi r}} \sin \frac{\theta}{2} \left(2 + \cos \frac{\theta}{2} \cos \frac{3\theta}{2}\right) + \sigma_\infty \cos 2\beta \quad (16)$$

$$\sigma_y = \frac{K_1}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2}\right) + \frac{K_{II}}{\sqrt{2\pi r}} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2} \quad (17)$$

$$\tau_{xy} = \frac{K_1}{\sqrt{2\pi r}} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2} + \frac{K_{II}}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2}\right) \quad (18)$$

where the stress intensity factors $K_I, K_{II}$ are given by:

$$K_I = \sigma_\infty \sqrt{\pi \alpha} \sin^2 \beta, \quad K_{II} = \sigma_\infty \sqrt{\pi \alpha} \sin \beta \cos \beta. \quad (19)$$

1) For singular solution

By substituting relations (16) to (19) into relation (12), we obtain:

$$\text{Det}(\sigma_{ij}) = D/2\pi r \quad (20)$$

where:

$$D = K_1^2 D_1 + K_{II}^2 D_2 + K_1 K_{II} D_{12} \quad (21)$$

with:

$$D_1 = \cos^4 \frac{\theta}{2} \quad (22)$$

$$D_2 = \cos^2 \frac{\theta}{2} \left(3 \sin^2 \frac{\theta}{2} - 1\right) \quad (23)$$

$$D_{12} = -4 \sin \frac{\theta}{2} \cos^3 \frac{\theta}{2}. \quad (24)$$

By substituting the stress intensity factors $K_I, K_{II}$ from relation (19) into relation (21), we obtain:

$$\frac{D}{\pi \alpha \sigma_\infty^2} = D_1 \sin^4 \beta + D_2 \sin^2 \beta \cos^3 \beta + D_{12} \sin^3 \beta \cos \beta. \quad (25)$$

Relation (25) gives the variation of the determinant of the stress tensor around the crack tip. Figure 2(a), (b) and (c) present the variation of the determinant of the stress tensor around the tip of the typical case of an internal crack obliqueness $\beta = 90^\circ, 30^\circ$, and $5^\circ$ respectively. In Fig. 2(a) the distribution of the determinant of the stress tensor presents a positive maximum at the position $\theta = 0^\circ$, while in Fig. 2(b) this distribution of the determinant of the stress tensor presents two
Fig. 2. The distribution of the determinant of the stress tensor around the tip of an internal crack obliqueness. (a) $\beta = 90^\circ$, (b) $\beta = 30^\circ$ and (c) $\beta = 5^\circ$.
positive maxima at the positions \( \theta = -83^\circ \) and \( +130^\circ \), respectively, and a negative maximum at the position \( \theta = +31^\circ \). According to relations (10) and (11) the \( T_V \) takes the maximum value, \( T_{V, \text{max}} \), for the positive maximum of the \( \text{Det.}(g_{ij}) \) at the position \( \theta = \theta_0 = 0^\circ \) (Fig. 2a) and \( \theta = \theta_0 = -83^\circ \) (Fig. 2b). Therefore, the crack will propagate to direction \( \theta_0 = 0^\circ \) and \( \theta_0 = -83^\circ \), respectively.

The position of the first positive maximum (\( \theta_0 \)) may be defined by annulling the partial derivative of the relation (21):

\[
\frac{\partial \text{Det.}(g_{ij})}{\partial \theta} = 0.
\]

(26)

Figure 3 presents the variation of \( -\theta_0 \) versus \( \beta \). In the same figure, the variation of \( -\theta_0 \), according to \( T-, G-, S- \) and \( S_2- \) criteria, was plotted.

The critical stress \( \sigma_{\infty}^\beta \) of crack initiation is calculated by the relations (15) and (20) to (25):

\[
\text{Det.}(\sigma_{ij})|_{\theta = \theta_0} = \frac{(\sigma_{\infty}^\beta)^2}{2\left(\frac{r}{\alpha}\right)\beta} (D_1 \sin^4 \beta + D_2 \sin^2 \beta \cos^2 \beta + D_{12} \sin^3 \beta \cos \beta)
\]

(27)

or

\[
\sigma_{\infty}^\beta|_{\theta = \theta_0} = \left[ \frac{2\left(\frac{r}{\alpha}\right)^\beta \text{Det.}(\sigma_{ij})_{\text{cr}}}{(D_1 \sin^4 \beta + D_2 \sin^2 \beta \cos^2 \beta + D_{12} \sin^3 \beta \cos \beta)} \right]^{1/2}.
\]

(28)

For \( \beta = 90^\circ \) and \( \theta = \theta_0 = 0^\circ \) the relation (28) becomes:

\[
\sigma_{\infty}^{90^\circ} = \left[ \frac{2\left(\frac{r}{\alpha}\right)^{90^\circ} \text{Det.}(\sigma_{ij})_{\text{cr}}}{(D_1 \sin^4 \beta + D_2 \sin^2 \beta \cos^2 \beta + D_{12} \sin^3 \beta \cos \beta)} \right]^{1/2}
\]

(29)

where \( \left(\frac{r}{\alpha}\right)^{\beta} \ll 1 \).

Fig. 3. Variation of \( -\theta_0 \) vs \( \beta \) for an internal oblique crack, according to \( \text{Det.-, } G(4)-, T[11]-, S[5]- \) and \( S_2(6)- \) criteria.
The ratio \((r/\alpha)^\beta\) depends on the angle \(\beta\) and defines the core-region around the crack tip. In static problems of cracks, the \((r/\alpha)^\beta\) is calculated by the initial curve of the caustics [19]:

\[
r^\beta = r^{90^\circ}(1 + \mu^2)^{1/5}
\]

where:

\[
\mu = \frac{K_{II}}{K_I} = \cot \beta.
\]

Then, relation (30) becomes:

\[
\left(\frac{r}{\alpha}\right)^\beta = \left(\frac{r}{\alpha}\right)^{90^\circ} \sin^{0.4} \beta.
\]

From relations (28), (29) and (32) we obtain:

\[
\frac{\sigma_\infty^\beta}{\sigma_\infty^{90^\circ}} \bigg|_{\alpha = \theta_0} = \sin^{0.2} \beta \frac{D_1 \sin^4 \beta + D_2 \sin^3 \beta \cos \beta + D_1 \sin^2 \beta \cos \beta}{(D_1 \sin^4 \beta + D_2 \sin^3 \beta \cos \beta + D_1 \sin^2 \beta \cos \beta)^{1/2}}
\]

Relation (33) gives the critical stress of crack initiation \(\sigma_\infty^\beta\) normalized to its characteristic value \(\sigma_\infty^{90^\circ}\) for \(\beta = 90^\circ\). Figure 4 presents the variation of the critical fracture stress versus the angle \(\beta\), according to Det.-criterion and \(\sigma_\alpha\), G-, S- and T-criteria.

It is worth-while to point out that the critical fracture stress could not take very high values, as it appears in Fig. 4, because there is an up-limit of critical fracture stress. This up-limit is the critical fracture stress of the uncracked plate, \(\sigma_{\infty}^{\text{uncr}}\). The relation between \(\sigma_\infty^{90^\circ}\) and \(\sigma_{\infty}^{\text{uncr}}\) is:

\[
\frac{\sigma_{\infty}^{\text{uncr}}}{\sigma_\infty^{90^\circ}} = \delta > 1.
\]

Fig. 4. Fracture stress \(\sigma_\infty^\beta\) vs \(\beta\), reduced to its values \(\sigma_\infty^{90^\circ}\) for \(\beta = 90^\circ\), according to Det.-, \(\sigma_\alpha[2]\), G[4]-, S[5]- and T[11]-criterion.
Then, from Fig. 4 we have that:

$$\sigma_{\infty}^{90^\circ} \geq \sigma_{\infty}^\theta < \sigma_{\infty}^{uncr}$$  \hspace{1cm} (35)

or

$$1 \leq \frac{\sigma_{\infty}^\theta}{\sigma_{\infty}^{90^\circ}} < \delta, \quad \text{for} \quad 0 < \theta \leq 90^\circ.$$  \hspace{1cm} (36)

From relation (36), we can conclude that the relation (33) can not take values higher than the ratio $\delta$ because if this ratio takes higher values of $\delta$ the cracked plate is possible to fracture elsewhere except the crack tip. So, the curves of Fig. 4 are limited by the ratio $\delta$.

(2) For solution with constant term

Respective relations to the relations (25) and (33) for the case of solution with the constant term of the stress $\sigma_x$ we can obtain:

$$D_c = \frac{D}{4 \pi \alpha \sigma_{\infty}^{90^\circ}} = D_1 \sin^4 \beta + D_2 \sin^2 \beta \cos^2 \beta + D_{12} \sin^3 \beta \cos \beta$$

$$+ \left[2 \left(\frac{r}{\alpha}\right)^{90^\circ}\right]^{1/2} \cos 2 \beta \sin^{1.2} \beta \left[\sin \beta \cos \frac{\theta}{2} \left(1 + \sin \frac{\theta}{2} \sin \frac{3 \theta}{2}\right) + \cos \beta \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3 \theta}{2}\right]$$  \hspace{1cm} (37)

and

$$\frac{\sigma_{\infty}^\theta}{\sigma_{\infty}^{90^\circ}} = \left\{1 - \left[2 \left(\frac{r}{\alpha}\right)^{90^\circ}\right]^{1/2}\right\}^{1/2} \frac{D_{1/2}}{D_c} \sin^{0.2} \beta.$$  \hspace{1cm} (38)

Figure 5 presents the variation of $-\vartheta_0$ versus $\beta$ for various values of the ratio $(r/\alpha)^{90^\circ}$, while Fig. 6 presents the variation of the critical fracture stress versus $\beta$ for various values of the ratio $(r/\alpha)^{90^\circ}$, according to Det.-criterion.

![Fig. 5. Variation of $-\vartheta_0$ vs $\beta$ for an internal oblique crack, according to Det.-criterion for singular solution and for $(r/\alpha)^{90^\circ} = 0.001, 0.03, 0.08, 0.2, 0.3.$](image)
Third stress invariant

It is worth-while observing that the constant term of the stress field not influences considerably the direction $\theta_0$ of initiation of crack propagation (Fig. 5), while it strongly influences the critical fracture stress (Fig. 6). So, for various values of ratio $(r/a)^{90^\circ}$, that means for various types of materials, brittle or ductile, the critical fracture stress reduce considerably.

In Fig. 6 experimental points of fracture stresses for various materials were placed [16, 20, 21, 22].

**CONCLUSION**

From this study is concluded that the third stress invariant is the main factor regulating the mode of fracture.

The distribution of the determinant of the third stress invariant along a circle around the crack-tip presents positive and negative maxima. The crack will propagate along the direction of the positive maxima and mainly along the direction of the local maximum ($-\theta_0$) because to this direction the initiation of the crack takes place with the minimum of critical stress.

The critical stress of initiation depends strongly on the ratio $(r/a)^{90^\circ}$. So, when the ratio $(r/a)^{90^\circ}$ increases the critical stress decreases considerably. This means that if the core-region at the crack-tip is large (ductile materials) the critical stress will be very low.

**REFERENCES**


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