Transient SIF results for a cracked viscoelastic strip under concentrated impact loading – An integral-transform/function-theoretic approach

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Abstract

The diffraction of transient SH waves by a long crack in a strip of viscoelastic material is considered. These viscoelastic waves are generated by a pair of suddenly applied, equal but opposite concentrated anti-plane forces, which tend to separate the crack faces in a tearing mode. In this way, the present solution provides Green’s function for the more general case of spatially non-uniform loading of the cracked strip. The diffraction problem, which contains two characteristic lengths, is solved asymptotically by a function-theoretic method in conjunction with a numerical technique for Laplace-transform inversion. Attention is focused on the time-dependent stress intensity factor (SIF). Extensive numerical results for the SIF dependence upon the viscoelastic material parameters and the ratio of characteristic lengths are provided.

1. Introduction

An important class of problems within the frameworks of Wave Motion and Fracture Mechanics concerns the stress-wave diffraction by cracks having the form of narrow cuts or slits. Typical of such problems are those involving a planar crack in a nominally elastic body under the action of dynamically applied loads. When a wave disturbance reaches the crack edge, a complicated scattered field radiates out behind certain wavefronts. The governing equations are of hyperbolic type and, therefore, transient wave fields are anticipated. Such fields, of course, differ markedly from corresponding equilibrium (static) fields.

Investigations into this type of problems originated with the pioneering work of de Hoop [1] and have intensively continued through the efforts of, among others, Achenbach [2,3], Achenbach and Brock [4], Nilsson [5], Brock [6–9], Freund [10–11], Freund and Rice [12], Atkinson [13], Kundu [14], Jiang and Knowles [15], Georgiadis and co-workers [16–20], and Librescu and Shalev [21].

In problems of elastodynamic Fracture Mechanics, most of the attention is focused usually on the field behavior near the crack tip during a small time-interval immediately after the rapid application of loading. Indeed, the determination of the stress intensity factor (SIF) at the crack tip provides a useful quantitative information about the material resistance to the onset of brittle fracture in dynamically loaded structures [2,4,11]. In particular, studies among the above-mentioned ones indicated that the ratio of dynamic SIF (transient problem) to corresponding static

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SIF (same configuration and loading, the latter, instead, being applied slowly) departs from unity as the time varies, exhibiting a pattern of local maxima and minima. Dynamic SIF overshoots are then possible and it is interesting to determine these effects by an exact problem formulation and analysis.

In the present study, we deal with the problem of diffraction of anti-plane shear (SH) waves by a semi-infinite crack in a long strip made by a viscoelastic material. As Fig. 1 shows, a planar crack is situated at the mid-width plane of the strip and two equal but opposite forces act suddenly (with a Heaviside step time-dependence) along the crack faces at a distance \( L \) from the crack tip. The strip has a finite width \( 2b \) in the \( y \)-direction, it is of infinite extent in the \( x \)-direction, and it is enough thick in the direction normal to the \((x, y)\)-plane so that an anti-plane shear state prevails. In some contrast with idealized geometries involving a cracked infinite domain, the present geometry allows including the influence of boundaries on the diffraction field and, in this way, allows for a better modeling of practical situations. Indeed, the initially diffracted waves by the crack edge return after their reflection at the lateral boundaries \((-\infty < x < \infty, y = \pm b)\) and are re-diffracted at the crack tip, in addition to other initially reflected waves which are also diffracted when reaching the crack-edge discontinuity. This process continues until waves radiate away from the crack-tip region to \( x \to \pm \infty \) and a steady state is reached.

Because of the presence of the characteristic length \( L \), this problem lies within a difficult sub-class of elastodynamic (having field equations of the wave type) crack problems involving spatially non-uniform loadings. Within the usual integral-transform analysis of these, an exponential term with unbounded behavior at infinity of a certain half-plane arises in the two-sided (spatial) Laplace (or Fourier) transform plane. So far, only a few specific problems have been solved [7,8,10–12,20,22]. Among these, only the studies of [20,22] consider a finite domain in the form of a strip. Here, the work in [20] is generalized by including viscoelastic effects, and in addition, more extensive numerical results are presented. In particular, our objective is the study of the influence of the various viscoelastic-material parameters and the ratio \((L/b)\) of the characteristic lengths upon the transient SIF. A general linear-viscoelastic material is considered in formulating the problem, but numerical results are presented only for a three parameter model (standard linear solid). However, as is well-known (e.g. [23]), the latter model represents realistically material response of many polymers and metals under stress-wave conditions. We note, in passing, the increasing usage of viscoelasticity theory in recent studies aiming at mechanical design (e.g. [24,25]) which certainly justifies analyses of viscoelastodynamic problems.

Here, some standard results of analytic-function theory (contour integration, Cauchy's integral formula, Jordan's lemma, Abel–Tauber theorems) are utilized to obtain an analytical closed-form solution for the crack-tip stress field in the time Laplace-transform domain. An alternative approach for the present problem could be the Wiener–Hopf (W–H) technique [3,26,27] However, it appears that the former approach (see also [20,28]) may also work in more

![Fig. 1. Sudden application of concentrated loads on the faces of a crack in a viscoelastic strip under anti-plane shear conditions.](image)
general cases where an equation containing the two unknown functions (which are to be decoupled) and the previously mentioned exponential forcing term holds only along a line in the complex plane and not necessarily over a strip. The latter statement can be realized in view of condition (9.49) in Achenbach’s treatise [3] which is strictly necessary for a sum splitting along a straight infinite line parallel to the imaginary axis and which precludes dealing with an exponential forcing term, and also in view of Plemelj formulas (see e.g. [3, Eqs. (9.54), (9.58) and (9.59)]), which inevitably will contain the aforementioned exponential term behaving as an integral function unbounded at infinity.

Our transformed solution is too complicated for an exact inversion. The latter task, instead, is accomplished numerically by a very efficient technique known as the DAC technique [29,30], which has extensively been employed by Georgiadis and co-workers [16–18,20,31] in a variety of elastodynamic diffraction problems. This technique, which is based on Fourier series and pertinent discretizations of real integrals, was also highly recommended by the well-known survey studies on Laplace-transform inversion of Davies and Martin [32] and Narayanan and Beskos [33].

2. Problem statement

As Fig. 1 depicts and already mentioned in Section 1, with respect to an Ox y z Cartesian coordinate system attached to the cracked body and having its origin at the crack tip, the strip occupies the region \((-\infty < x < \infty, -b < y < b)\) and is thick enough in the z-direction to allow a state of anti-plane shear. The crack is situated along the plane \((-\infty < x < 0, y = 0)\) and is sheared by a pair of anti-plane concentrated forces \(\pm F H(t)\), independent on the z-coordinate and acting along \((x = -L, y = 0)\). Because of anti-symmetry with respect to the plane \(y = 0\), the problem can be viewed as a half-strip problem for the region \((-\infty < x < \infty, 0 < y < b)\).

The governing equations for such a state of a general linearly viscoelastic material are written as (e.g. [34])

\[
\begin{align*}
\sigma_x, \sigma_y, \sigma_z, \tau_{xy} &= 0, \quad P[\tau_{yz}] = Q \left\{ \frac{\partial u_z}{\partial x} \right\}, \quad P[\tau_{yz}] = Q \left\{ \frac{\partial u_z}{\partial y} \right\}, \\
Q \left\{ \frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} \right\} &= \rho P \left\{ \frac{\partial^2 u_z}{\partial t^2} \right\},
\end{align*}
\tag{2a,b,c}
\]

where \((u_x, u_y, u_z)\) and \((\sigma_x, \sigma_y, \tau_{xy})\) are the components of the displacement vector and stress tensor, respectively, polynomials \(P\) and \(Q\) are differential operators defined as \(P[\cdot] \equiv \sum_{k=0}^{m} a_k (\partial^k [\cdot] / \partial t^k)\) and \(Q[\cdot] \equiv \sum_{k=0}^{n} b_k (\partial^k [\cdot] / \partial t^k)\), where \(a_k\) and \(b_k\) are material constants, and the integers \(m\) and \(n\) are characteristic quantities of each particular viscoelastic model. Also, \(\rho\) is the mass density of the material. Eqs. (2b,c) are the material constitutive equations, whereas (3) is the field equation. In the pure-elastic case, the operator

\[
c \equiv \left[ \frac{\mu(D)}{\rho} \right]^{1/2} \text{ with } D \equiv \frac{\partial}{\partial t} \text{ and } \mu = \frac{Q}{P},
\tag{4}
\]

degenerates into the shear wave velocity.

Furthermore, the associated initial/boundary value problem must satisfy the following conditions:

(i) Initial and boundary conditions :

\[
u_z(x, y, t) = \frac{\partial u_z(x, y, t)}{\partial t} = 0 \quad \text{for } t \leq 0,
\tag{5a}
\]

\[
\sum_{k=M}^{N} a_k \frac{\partial^{k-M} \tau_{yz}(0)}{\partial t^{k-M}} = \sum_{k=M}^{N} b_k \frac{\partial^{k-M} (\partial u_z(0)/\partial y)}{\partial t^{k-M}} \quad \text{with } M = 1, 2, \ldots, N \text{ and } \gamma = x, y,
\tag{5b}
\]

\[
\tau_{yz}(x, b, t) = 0 \quad \text{for } -\infty < x < \infty.
\tag{6}
\]
\[ \tau_{yz}(x, 0, t) = F \delta(x + L) H(t) \quad \text{for} \quad -\infty < x < 0, \]  
\[ u_z(x, 0, t) = 0 \quad \text{for} \quad 0 < x < \infty, \]  
which is equal to the larger one of \( m \) and \( n \) integers defined before, with some of the coefficients \( a_k \) and \( b_k \) set equal to zero to eliminate any terms which may absorb from the polynomials \( P \) and \( Q \). Also, \( \tau_{yz}(0), \frac{\partial u_z(0)}{\partial y}, \frac{\partial \tau_{yz}(0)}{\partial t}, \text{etc.} \), are the initial values of \( \tau_{yz}, \frac{\partial u_z}{\partial y}, \frac{\partial \tau_{yz}}{\partial t}, \text{etc.} \). We should notice that the additional initial conditions (5b) stem from the particular constitutive law considered here (viz. Eqs. (2b,c)), and allow for the possibility of a “sudden” solution of the problem (cf. [34, pp. 416–423]). In Eq. (7), \( \delta(\cdot) \) is the Dirac delta distribution, and \( H(\cdot) \) is the Heaviside unit-step function.

(ii) Edge conditions:
\[ \tau_{yz}(x, 0, t) = O(x^{-\alpha}) \quad \text{with} \quad \alpha < 1 \quad \text{for} \quad x \to 0^+, \]  
\[ u_z(x, 0, t) = O(1) \quad \text{for} \quad x \to 0^-, \]  
which guarantee that the near-tip stress and displacement fields will not be so singular as to correspond to sources of radiated energy. Integreal kinetic and potential energy is, in turn, a necessary condition for solution uniqueness (e.g. [3]). Furthermore, on the basis of Fracture Mechanics considerations or by exact asymptotic analysis (see e.g. [11,35–37]), it can be shown that \( \tau_{yz}(x, 0, t) \sim x^{-1/2} \) for \( x \to 0^+ \), and \( u_z(x, 0, t) \sim x^{1/2} \) for \( x \to 0^- \). We note, however, that (9) and (10) are still sufficient conditions for applying Jordan’s lemma in subsequent steps of our analysis, although we rather chose to rely on the former (restricted) asymptotic behavior.

(iii) Finiteness conditions at remote regions:
\[ |\bar{\tau}_{yz}(x, 0, s)| < A \exp(-p_T x) \quad \text{for} \quad x \to +\infty, \]  
\[ |\bar{u}_z(x, 0, s)| < B \exp(p_U x) \quad \text{for} \quad x \to -\infty, \]  
where \( A, B, p_T \) and \( p_U \) are positive quantities, an overbar denotes the time Laplace transform of a function (as defined below) and \( s \) is the corresponding variable. These equations are a direct consequence of the asymptotic behavior of the time Laplace-transformed Green’s function of the wave equation (3) and guarantee that the diffraction field at infinity consists of outgoing waves only.

In what follows, the problem defined by (1)–(12) will be solved asymptotically. In particular, the stress field near to the crack edge, \( \tau_{yz}(x \to 0^+, 0, t) \), will be obtained by a combination of exact and numerical analysis.

3. Basic analysis

As a first step, the following one- and two-sided Laplace transform (LT) pairs are introduced in order to suppress the \( t \)- and \( x \)-dependence, respectively, in the governing equations and the initial/boundary conditions
\[ \tilde{f}(x, y, s) = \int_0^{\infty} f(x, y, t) e^{-st} \, dt, \quad f(x, y, t) = \frac{1}{2\pi i} \int_{Br} \tilde{f}(x, y, s) e^{st} \, ds, \]  
\[ \tilde{f}^*(p, y, s) = \int_{-\infty}^{+\infty} f(x, y, s)e^{-sp} \, dx, \quad \tilde{f}(x, y, t) = \frac{s}{2\pi i} \int_{Br} \tilde{f}^*(p, y, s)e^{sp} \, dp, \]  
where the function \( f(x, y, t) \) is assumed to be Laplace transformable on \( x \) and \( t \), and \( Br \) denotes the Bromwitch path in pertinent complex planes.
In addition, half-line spatial transforms of the unknown functions $\tilde{r}_{yz}(x, 0, s)$, $0 < x < \infty$, and $\tilde{u}_z(x, 0, s)$, $-\infty < x < 0$, are also defined

$$T^+(p, s) = \int_0^\infty \tilde{r}_{yz}(x, 0, s) e^{-spx} \, dx \quad \text{for} \quad -(p_T/s) < \Re(p),\quad (15a)$$

$$\tilde{r}_{yz}(x, 0, s) = \frac{1}{2\pi i} \int_{Br_1} T^+(p, s) e^{spx} \, dp \quad \text{for} \quad 0 < x < \infty,\quad (15b)$$

$$U^-(p, s) = \int_{-\infty}^0 \tilde{u}_z(x, 0, s) e^{-spx} \, dx \quad \text{for} \quad \Re(p) < (p_U/s),\quad (16a)$$

$$\tilde{u}_z(x, 0, s) = \frac{1}{2\pi i} \int_{Br_2} U^-(p, s) e^{spx} \, dp \quad \text{for} \quad -\infty < x < 0,\quad (16b)$$

where, in light of conditions (11) and (12), $T^+(p, s)$ and $U^-(p, s)$ are analytic functions in the right, $-(p_T/s) < \Re(p)$, and left, $\Re(p) < (p_U/s)$, half-plane, respectively.

Now, applying (13a) and (14a) successively to the field equation (3) gives, in view of (5), an ordinary differential equation with the general solution

$$\tilde{u}_z^*(p, y, s) = C(p, s) \exp(-s[(\tilde{c})^{-2} - p^2]^{1/2} y) + D(p, s) \exp(s[(\tilde{c})^{-2} - p^2]^{1/2} y),\quad (17)$$

where $C(p, s)$ and $D(p, s)$ are unknown functions, and

$$\tilde{c}(s) = \left[ \frac{\tilde{\mu}(s)}{\rho} \right]^{1/2} \quad \text{with} \quad \tilde{\mu}(s) = \frac{\tilde{Q}(s)}{P(s)}.\quad (18)$$

At this point, it should be noted that for the partial differential equation (3) to remain hyperbolic, the limit of the effective wave speed $\tilde{c}(s)$ as $s \to \infty$ must be finite and positive. Further, by applying (13a) and (14b) to (5)–(8), taking into account (15a) and (16a), and eliminating $C(p, s)$ and $D(p, s)$ from the resulting system of three equations in four unknown functions, the following functional equation is obtained:

$$T^+(p, s) + Fe^{sp}/s = -\tilde{\mu}(s) K(p, s) U^-(p, s),\quad (19)$$

where the kernel is given as

$$K(p, s) = s[(\tilde{c})^{-2} - p^2]^{1/2} \tanh(s[(\tilde{c})^{-2} - p^2]^{1/2} b),\quad (20)$$

whereas, in view of (15a) and (16a), we notice that (19) holds over the strip of analyticity, $-(p_T/s) < \Re(p) < (p_U/s)$.

The main task of the analysis is to produce separate equations for both the yet-unknown functions $T^+(p, s)$ and $U^-(p, s)$, which enter the single equation (19), through a decoupling procedure. This is possible by supplying (19) with results obtained by using analytic-function theory.

To proceed further we need to perform a product factorization of the kernel $K(p, s)$. By employing the infinite-product form of the pertinent hyperbolic trigonometric function (see e.g. [38, p.85]), one finds

$$K(p, s) = b x^2 [(\tilde{c})^{-2} - p^2] \prod_{k=1}^{\infty} \left[ \frac{(2k - 1)^2 (p + A_k)(p - A_k)}{4 k^2 (p + B_k)(p - B_k)} \right].\quad (21)$$
where

\[
A_k = \left[ \left( \frac{k \pi}{bs} \right)^2 + (\tilde{c})^{-2} \right]^{1/2}, \quad B_k = \left[ \left( \frac{(2k-1) \pi}{2bs} \right)^2 + (\tilde{c})^{-2} \right]^{1/2},
\]

the latter being real and positive quantities for \( s \) considered a real and positive parameter. Then, a product factorization follows easily from (21) as

\[
K(p, s) = K^+(p, s)K^-(p, s),
\]

where

\[
K^+(p, s) = b^{1/2} s [((\tilde{c})^{-1} + p) \prod_{k=1}^{\infty} \left( \frac{(2k-1)(p + A_k)}{2k(p + B_k)} \right)], \quad K^-(p, s) = K^+(-p, s),
\]

and the functions \( K^+(p, s) \) and \( K^-(p, s) \) are non-zero and analytic in \( \text{Re}(p) > - \inf(A_{k=1}, B_{k=1}) \) and \( \text{Re}(p) < \inf(A_{k=1}, B_{k=1}) \), respectively.

We now proceed to some re-arrangements in (19). Indeed, in view of (23), Eq. (19) can be written as

\[
\frac{T^+(p, s)}{K^+(p, s)} + \frac{F e^{-is\omega}}{s K^+(p, s)} = -\tilde{\mu}(s) K^-(p, s) U^-(p, s),
\]

which holds over the strip \(-\inf(\omega_T/s, A_{k=1}, B_{k=1}) < \text{Re}(p) < \inf(\omega_U/s, A_{k=1}, B_{k=1})\).

Further, another crucial step in the decoupling procedure is changing the variable from \( p \) to \( \omega \) and dividing both sides of (25) by \( 2\pi i (\omega - p) \), thus having

\[
\frac{T^+(\omega, s)}{2\pi i K^+(\omega, s)(\omega - p)} + \frac{F e^{-is\omega}}{2\pi i s K^+(\omega, s)(\omega - p)} = -\frac{\tilde{\mu}(s)}{2\pi i} \frac{K^-(\omega, s) U^-(\omega, s)}{(\omega - p)}. \tag{26}
\]

Then, we integrate (26) over the imaginary axis \( \text{Im}(\omega) \), by taking also the point \( p \) to lie on the right half-plane \( \text{Re}(\omega) > 0 \) only

\[
\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{T^+(\omega, s)}{K^+(\omega, s)(\omega - p)} d\omega + \frac{F}{2\pi i s} \int_{-\infty}^{\infty} \frac{e^{-is\omega}}{K^+(\omega, s)(\omega - p)} d\omega = -\frac{\tilde{\mu}(s)}{2\pi i} \int_{-\infty}^{\infty} \frac{K^-(\omega, s) U^-(\omega, s)}{(\omega - p)} d\omega. \tag{27}
\]

Now, working on the integral in the r.h.s. of (27), we observe that \( K^-(\omega) \sim \omega^{1/2} \) for \(|\omega| \to \infty\), \( U^-(\omega, s) \sim \omega^{-3/2} \) for \(|\omega| \to \infty\) (the edge condition that \( u_2(x, 0, t) \sim x^{1/2} \) for \( x \to 0^- \) was utilized in conjunction with the Abel-Tauber theorem [39]) and also that the integrand is an analytic function in the left half-plane \( \text{Re}(\omega) < 0 \). Therefore, by applying contour integration with deforming the integration path to include a large semi-circle at infinity in the left half-plane, as Fig. 2 shows, and Jordan’s lemma [39] to this integral, we conclude by Cauchy’s theorem that it vanishes when \( p \) is a point of the right half-plane \( \text{Re}(\omega) > 0 \).

Then by employing again contour integration and Jordan’s lemma on the first integral of the l.h.s. of (27), and by closing the integration path with a large semi-circle at infinity in the right half-plane, we obtain the following result through Cauchy’s integral formula:

\[
\frac{T^+(p, s)}{K^+(p, s)} = -\frac{F}{2\pi i s} \int_{-\infty}^{\infty} \frac{e^{-is\omega}}{K^+(\omega, s)(\omega - p)} d\omega \quad (p \text{ in } \text{Re}(\omega) > 0). \tag{28}
\]
Finally, the integral in (28) can formally be evaluated by closing the integration path with a large semi-circle at infinity in the left half-plane (Fig. 2). Again, use of Jordan’s lemma and Cauchy’s residue theorem lead to the following expression for the transformed crack-line stress:

\[ T^+(p, s) = -\frac{F}{b^{1/2} s^{2}} \cdot K^+(p, s) \left( \left[ \sum_{j=1}^{\infty} \left[ \frac{\omega + A_j}{\omega + (\tilde{c})^{-1}} \cdot e^{L_{1\omega}} \left( \prod_{k=1}^{\infty} \left( \frac{2k}{2k - 1} \cdot \frac{\omega + B_k}{\omega + A_k} \right) \right) \right] \right]_{\omega = -A_j} \]

\[ + \left[ \prod_{k=1}^{\infty} \left( \frac{2k}{2k - 1} \cdot \frac{- (\tilde{c})^{-1} + B_k}{-(\tilde{c})^{-1} + A_k} \right) \right] \frac{e^{-L_{1\omega} \tilde{c}}}{-(\tilde{c})^{-1} - p} \times \left( \sum_{j=1}^{\infty} \right) \]

(29)

where (24a) was also employed, and \( A_j \) are given in (22) but with \( j \) replacing \( k \).

At this point, it should be noted that one could alternatively arrive at Eq. (28) by employing the standard Wiener-Hopf technique [3,26,27] and performing an additive decomposition of the term \( [(Fe^{L_{1\omega}})(sK^+(p, s))^{-1}] \) inside the strip \( -\inf(p_T/s, A_k = 1, B_k = 1) < \Re(p) < -\inf(p_U/s, A_k = 1, B_k = 1) \) according to [27, Theorem B, p. 13]. However, if Eq. (25) were to hold along a line, say \( \Re(p) = 0 \), such a decomposition could not be possible because the necessary condition (9.49) in [3, p. 369] would be violated by the term \( e^{L_{1\omega}} \). Furthermore, a direct sum splitting of the term \( [(Fe^{L_{1\omega}})(sK^+(p, s))^{-1}] \) along \( \Re(p) = 0 \), according to the pertinent Plemelj formula would result to the appearance of the term \( e^{L_{1\omega}} \) outside the Cauchy principal-value integral (similar to the one in the r.h.s. of (28) above) which, in turn, inhibits the use of Liouville’s theorem.

Eq. (29) may provide, through the two LT inversions in (13b) and (14b), the crack-line stress, \( \tau_{yz}(x, 0, t) \). Since, however, exact and analytical inversions in (29) are impossible, we shall resort to asymptotics and numerics for returning to the original space/time domain. In particular, an asymptotic analytical inversion according to (14b) will give \( \tilde{\tau}_{yz}(x \to 0^+, 0, s) \), and then, a numerical inversion (via the DAC technique) for (13b) will provide values of \( \tau_{yz}(x \to 0^+, 0, t) \) and its intensity at the crack edge. Still, these asymptotic results constitute all the necessary information about the diffraction field from the perspective of elastodynamic Fracture Mechanics [4,11], where the quantity of main interest is the stress-intensity-factor defined within the present problem as

\[ K_{III} = \lim_{x \to 0^+} \left[ (2\pi x)^{1/2} \cdot \tau_{yz}(x, 0, t) \right] \]

(30)
and which is a function of loading \((F)\), geometry \((L/b)\), material parameters \((c)\), and time \(t\). The subscript “III” denotes, according to the standard Fracture Mechanics notation, that the crack is in an anti-plane shear (mode III) stress environment.

4. Singular crack-line stress in the one-sided LT domain

The Abel–Tauber theorem [39], which relates asymptotically original functions and their transforms, is utilized to calculate the singular part of stress, \(\lim_{x \to 0^+} \bar{\tau}_{yx}(x, 0, s)\), from the large-\(p\) expression of (29), \(\lim_{p \to \infty} T^+(p, s)\).

First, an asymptotic expression for the kernel in (20) is obtained as

\[
\lim_{p \to \infty} K(p, s) = -isp,
\]

where \(p\) is taken along the pertinent Bromwich path. Then, on rewriting (31) under the form

\[
\lim_{p \to \infty} K(p, s) = -s(\epsilon^2 - p^2)^{1/2}, \quad \text{with} \quad \epsilon \to 0,
\]

the asymptotic kernel factorization follows easily as

\[
\lim_{p \to \infty} K^+(p, s) = (sp)^{1/2}, \quad \lim_{p \to \infty} K^-(p, s) = -s^{1/2}(-p)^{1/2}.
\]

On the other hand, the terms \((\omega - p)^{-1}\) and \([(\bar{c})^{-1} - p]^{-1}\) in (29) are approximated by \((-p)^{-1}\).

The above considerations lead to the following large-\(p\) approximation of (29):

\[
\lim_{p \to \infty} T^+(p, s) = \frac{F}{b^{1/2} s} \frac{1}{(sp)^{1/2}} \left( \left[ \sum_{j=1}^{\infty} \left[ \frac{\omega + A_j}{\omega + (\bar{c})^{-1}} e^{L_s \omega} \prod_{k=1}^{\infty} \left( \frac{2k}{2k - 1} \frac{\omega + B_k}{\omega + A_k} \right) \right] \right]_{\omega = -A_j} + \left[ \prod_{k=1}^{\infty} \left( \frac{2k}{2k - 1} \frac{-(\bar{c})^{-1} + B_k}{-(\bar{c})^{-1} + A_k} \right) e^{-L_s \bar{c}} \right],
\]

whose inverse follows from the definition (14b) and the Abel–Tauber theorem as

\[
\lim_{x \to 0^+} \bar{\tau}_{yx}(x, 0, s) = \frac{F}{(\pi bx)^{1/2} s} (\cdots),
\]

and the expression in the parentheses is the same as that in (34). The latter expression is, of course, only \(s\)-dependent, and thus, spatially independent.

Finally, from the definition of the SIF in (30) and employing (35), the Laplace-transformed SIF is found as

\[
\tilde{K}_{\text{III}} = F \left( \frac{2}{b} \right)^{1/2} \frac{1}{s} (\cdots),
\]

where, again, \((\cdots)\) is given in (34). Clearly, Eq. (36) provides a closed-form exact formula for the Laplace-transformed SIF. The infinite series and products are convergent, whereas a check on our method and results can be made by recovering to the corresponding static and pure-elastic problem from (36). Indeed, by considering an infinite shear-wave speed or a zero frequency \((c^{-1} = 0 \text{ in (3)})\) and \(s = 1\), (36) degenerates to the static result

\[
K_{\text{III}}^{\text{static}} = F \left( \frac{2}{b} \right)^{1/2} \left[ 1 + \frac{1}{2} e^{-\pi L/b} + \frac{3}{8} e^{-2\pi L/b} + \frac{15}{48} e^{-3\pi L/b} + \cdots \right],
\]
which is identical to the one obtained by Sih [40] through the use of conformal mapping and complex-variable analysis.

The remaining task now is the one-sided LT inversion of (36). Section 5 discusses how this can be made possible by numerical means.

5. Numerical inversion of the one-sided LT

Previous experience with numerical LT inversions (see e.g. 16–18,20,31, and survey articles 32,33) suggests the DAC technique [29,30] as a powerful means for inverting (36) according to the operation (13b). Of course, with such a numerical technique, one may obtain reliable results only for a limited time-interval and not for the whole time domain. This is due to the inevitable instability of the first-kind integral equation for \( f(t) \) – see Eq. (13a). Nevertheless, we proceed in this way because we are only interested in obtaining the SIF history during a small time-interval soon after the application of the impact loads and the ensuing wave diffraction at the crack tip. It is this time-interval during which interesting dynamic SIF overshoots occur, before energy is transferred through the wave motions to regions far from the crack edge and a steady state is reached.

The DAC technique is briefly outlined here. One may start from (13b) and write the following alternative form of the original function \( f(t) \):

\[
    f(t) = \frac{e^{ut}}{\pi} \int_{0}^{\infty} \left[ \text{Re} \tilde{f}(w + iu) \cos(ut) - \text{Im} \tilde{f}(w + iu) \sin(ut) \right] du, \tag{38}
\]

where \( s = w + iu \). Application of the trapezoidal rule for integrals over semi-infinite intervals leads now to an approximate expression for \( f(t) \) in the form of a Fourier series [41]:

\[
    f(t) \approx \left( \frac{e^{ut}}{T} \right) \left[ \frac{1}{2} \tilde{f}(w) + \sum_{k=1}^{\infty} \left[ \text{Re} \tilde{f}(w + ik\pi/T) \cos(k\pi t/T) - \text{Im} \tilde{f}(w + ik\pi/T) \sin(k\pi t/T) \right] \right]. \tag{39}
\]

Crump [30] has presented a systematic error analysis in the above procedure, from which \( f(t) \) can be computed to a predetermined accuracy. First, \( T \) is chosen so that \( 2T > t_{\text{max}} \) (where \( t_{\text{max}} \) is the time up to which results are to be obtained) and then \( w \) is determined by \( w = q - [\text{Ln}(E)]/2T \), where \( q \) is a number slightly larger than \( \text{max} [\text{Re}(s_0)] \) (where \( s_0 \) is a pole of \( \tilde{f}(s) \)) and the relative error is to be no greater than \( E \) (\( E = 10^{-8} \) was used in our computations).

It is possible also to increase the rate of convergence of (39) and thus reduce the truncation error by using a suitable transformation like the \textit{epsilon algorithm}

\[
    \epsilon_{n+1}^{m} = \epsilon_{n-1}^{(m+1)} + [\epsilon_{n}^{(m+1)} - \epsilon_{n}^{(m)}]^{-1} \text{ with } \epsilon_{-1}^{m} = 0, \tag{40}
\]

where \( \epsilon_{n}^{(m)} \) is the \( m \)th partial sum of (39). This accelerating algorithm was utilized here as in [16–18,20,31] as well.

Finally, it should be mentioned that Georgiadis [18] has presented extensive checks on the DAC technique and comparisons with other numerical techniques for one-sided LT inversion. His results demonstrate the overall efficacy of the technique and, particularly, its suitability for elastodynamic problems of the type discussed here.

6. Numerical procedure and results

All results were obtained by taking 20 terms of the series in (34), 1000 terms of the products in (34), and 250 terms of the Fourier series in (39) along with the accelerating epsilon algorithm. We note that the infinite series
and products in (34) seemed to have a rapid convergence showing little variation after their 5th and 10th term, respectively. Since, however, the LT inversion scheme is very sensitive upon even small inaccuracies (noise) in the values of the transformed function, we were enough cautious to obtain very accurate results for $\tilde{K}_{III}(s)$ by considering many terms in the series and products of (34). The computation time on a PC 486 for obtaining a SIF graph was typically 6 min.

On the other hand, an indication that a numerical LT inversion of $\tilde{K}_{III}(s)$ is feasible, since this function behaves in a monotonic and smooth fashion, appears in a typical graph (see Fig. 3) of the normalized transformed SIF $[\tilde{K}_{III}/F(2/b)^{1/2}]$ versus $(c/bs)$, where the LT variable $s$ was taken as a real and positive variable and $c$ was taken to represent rather a constant (i.e. a dynamical but pure-elastic state was assumed, where $c$ is the SH wave velocity) than an operator. Two different ratios $L/b = 1.0$ and 3.0 were considered. One could observe in Fig. 3 that: (a) for large $s$, viz. small $t$, the SIF is zero, which is to be expected since no wavefront reaches the crack tip up to the time $t = L/c$, and (b) for small $s$, viz. large $t$, the SIF tends to a plateau, which corresponds to the steady state in the time domain.

Now, results for the viscoelastic problem (here, $c$ is an operator) in the time domain will be presented. In particular, numerical results for $K_{III}(t)$ were obtained for the standard–linear–solid viscoelastic model characterized by the following operators ratio:

$$\mu(D) = \mu_{ST} \frac{\partial/\partial t + 1/\tau}{\partial/\partial t + (1 + f)/\tau}, \quad (41)$$

where $f = [(\mu_{ST}/\mu_{LT}) - 1]$ is a measure of the difference between the short-time $\mu_{ST}$ and the long-time $\mu_{LT}$ shear moduli, and $(\tau/(1 + f))$ is the relaxation time [18,34]. Application of the one-sided LT on (41) yields the following form of the “Lamé function” entering (34) through $\tilde{c}(s); \mu(s) = \mu_{ST}[s + (1/\tau)][s + ((1 + f)/\tau)]^{-1}$. In the present study, a material model was used with constant $\mu_{ST} = 1728$ MPa and density $\rho = 1200$ kg m$^{-3}$, and

![Fig. 3. Normalized transformed SIF $[K_{III}/F(2/b)^{1/2}]$ versus $(c/bs)$, where $s$ is taken as a real and positive variable.](image-url)
variable \( f, \tau \) in each case. These groups of material constants were considered in order to examine the effect of the viscoelastic behavior upon SIF. Notice that, in view of (41), the pure-elastic case can be recovered from this material model by letting either \( f \to 0 \) or \( \tau \to \infty \). We should also mention that the numerical values of \( f \) and \( \tau \) utilized for obtaining our results fall into the range of values found in impact experiments employing PMMA \([42]\) and other polymers \([43]\), and therefore, the present material model is a realistic one.

Regarding now the suitability of the expression in (36) for a LT inversion by means of a Bromwich-path integral (i.e. analyticity of the r.h.s. of (36) over a certain right half-plane of the LT complex variable \( s \)), an inspection leads to the conclusion that the right half-plane \( \text{Re}(s) > 0 \) is free from poles and branch cuts for the function \( \hat{K}_{\text{III}}(s) \), and therefore, the LT inversion of (36) can be obtained, in principle, by the procedure described in Section 5.

In all graphs which will be presented below, normalized quantities appear. We employ:

(a) a normalized stress intensity factor defined as \( [K_{\text{III}}/F(2/b)]^{1/2} \), (b) a dimensionless time defined as \( (c_{\text{ST}}t/b) \), where \( c_{\text{ST}} \equiv (\mu_{\text{ST}}/\rho)^{1/2} \) is the short-time shear wave velocity whose numerical value for our material model is 1200 m/s, and (c) a dimensionless relaxation time defined as \( g \equiv (c_{\text{ST}}\tau/b) \).

Figs. 4–6 show the normalized SIF versus dimensionless time for the constant ratio of characteristic lengths \( L/b = 1.0 \). The quantity \( f \) is kept constant in each case depicted in the respective graph, but variable dimensionless relaxation time \( g = 0.05, 2.00 \) and 10.00 is considered.

In Fig. 4, the combination of the small \( f \) (\( f = 0.15 \)) with a medium or large \( g \) (\( g = 2.00 \) or 10.00) yields a material response that is very close to purely elastic. One could observe that:

(a) \( K_{\text{III}}(t) \to 0 \) for \( (c_{\text{ST}}t/b) < 1 \) because no wavefront has arrived yet at the crack tip, (b) there is a step-type increase of the transient SIF, up to the value taken by the static SIF in a body of infinite extent containing a semi-infinite crack sheared by point forces, once the initial wavefront emitted at \( (x = -L, y = 0, t = 0) \) reaches the crack tip, (c) the latter value of the transient SIF remains steady until the first reflection of the initial SH wavefront on the lateral strip faces \( y = \pm b \) reaches the crack tip, then a step-type increase of SIF occurs again up to the value corresponding to the steady-state problem of a cracked strip, (d) afterwards, several reflections of the diffracted field contribute more or less severe \( K_{\text{III}}(t) \) variations until a steady-state is reached, (e) strong dynamic effects are pronounced for a time interval of about \( (c_{\text{ST}}t/b) < 7.0 \), and (f) there is a dynamic SIF overshoot on the order of 22% with respect to the steady-state value, which is \( \sim F(2/b)\).

We should also note that the SIF variation for the time-interval before the arrival of the first reflection of the initial wavefront, i.e. before any wave information reaches the crack tip from the strip boundaries, agrees fairly well with

![Fig. 4. Normalized SIF \([K_{\text{III}}/F(2/b)]^{1/2}\) versus dimensionless time \((c_{\text{ST}}t/b)\), for \( L/b = 1.0, f = 0.15 \) and several values of the dimensionless relaxation time \( g \).](image-url)
Fig. 5. Normalized SIF \( \left[ K_{\text{II}} / F(2/b)^{1/2} \right] \) versus dimensionless time \( (c_{ST}t/b) \), for \( L/b = 1.0 \), \( f = 2.00 \) and several values of the dimensionless relaxation time \( g \).

Fig. 6. Normalized SIF \( \left[ K_{\text{II}} / F(2/b)^{1/2} \right] \) versus dimensionless time \( (c_{ST}t/b) \), for \( L/b = 1.0 \), \( f = 10.00 \) and several values of the dimensionless relaxation time \( g \).

the corresponding analytical result obtained by Ma and Chen [44] for the cracked infinite plane.

In Fig. 4 again, when the relaxation time is small \( (g = 0.05 \) – solid line in the graph), there are still some viscous effects in the SIF variation, but these are restricted because of the small \( f \) considered. In particular, one could observe here the retarded and not-too-sharp variations as compared to the ones in the previous cases of almost purely elastic response. This is due to the fact that the viscoelastic wavefronts are not sharp, in contrast, of course, with the corresponding elastic wavefronts.

Fig. 5 illustrates SIF-time graphs for \( f = 2.00 \). Here, variations of the relaxation time produce pronounced viscoelastic effects because of the appreciable difference (expressed by the value of \( f \)) between the short- and long-time shear moduli. Only when the relaxation time is very large \( (g = 10.00) \), the SIF variation somewhat reminds the purely elastic case. On the other hand, it is noticed that the delayed elasticity of the material model exhibited
Fig. 7. Normalized SIF \( \frac{K_{III}}{F(2/b)^{1/2}} \) versus dimensionless time \( c_{ST}t/b \), for \( L/b = 1.0 \), \( g = 0.05 \) and several values of \( f \).

for very small relaxation time \( (g = 0.05) \). We refer to the studies by Georgiadis and co-workers [16–18,31] for an explanation to the occurrence of this viscoelastic phenomenon in crack and stress-concentration dynamic problems. Fig. 6 illustrates SIF-time graphs for \( f = 10.00 \), i.e. for a very pronounced viscous material behavior.

Figs. 7–9 present SIF-time graphs for \( L/b = 1.0 \), constant \( g \) in each case, and variable \( f \). Fig. 7 illustrates four graphs when the relaxation time is small \( (g = 0.05) \) and the difference between shear moduli takes small \( (f = 0.15) \), medium \( (f = 2.00) \) and large \( (f = 10.00) \) values. The purely elastic case \( (f = 0.00) \) is also depicted in the same figure. Figs. 8 and 9 illustrate also four graphs each for medium \( (g = 2.00) \) and large \( (g = 10.00) \) relaxation time. Interesting changes on the SIF history occur because of the changes in the material behavior. In some cases, there are no sharp variations, and, in some other cases, the SIF increases with time due to the retarded elasticity of the viscoelastic model.

Figs. 10 and 11 now present SIF-time graphs for \( L/b = 2.00 \), i.e. for a more distant point of application of the concentrated forces from the crack edge.

Fig. 8. Normalized SIF \( \frac{K_{III}}{F(2/b)^{1/2}} \) versus dimensionless time \( c_{ST}t/b \), for \( L/b = 1.0 \), \( g = 2.00 \) and several values of \( f \).
Fig. 9. Normalized SIF $[K_{III}/F(2/b)^{1/2}]$ versus dimensionless time $(c_{STM}/b)$, for $L/b = 1.0$, $g = 10.00$ and several values of $f$.

Fig. 10. Normalized SIF $[K_{III}/F(2/b)^{1/2}]$ versus dimensionless time $(c_{STM}/b)$, for $L/b = 2.0$, $f = 0.15$ and several values of $g$.

In Fig. 10, $f = 0.15$ is the constant value of the shear-moduli difference, whereas three different values of $g (= 0.05, 2.00$ and $10.00)$ are considered. Because $f$ is small, the material response is close to purely elastic, and therefore, similar SIF histories are observed here with those in Fig. 4. Appreciable overshoots (on the order of 22–30%) also occur with respect to the steady-state SIF value.

Fig. 11 illustrates four graphs for constant relaxation time and variable $f$. Because of the small $g (= 0.05)$, the material response is very sensitive to even small variations of $f$. The case $f = 0.00$ represents purely elastic response. Step-type variations clearly mark the initial and first-reflected wavefronts, and an overshoot of 32% is observed. This overshoot occurs because of the returning reflections at the lateral strip faces of diffracted wavefronts.

Finally, Fig. 12 shows three SIF-time graphs for $f = 0.15$ and $g = 0.05, 2.00$ and $10.00$. Because of the small $f$ considered, the SIF history resembles the one for the purely elastic case when medium ($g = 2.00$) and large ($g = 10.00$) relaxation time is assumed. Also, the SIF dynamic overshoot is still appreciable in these cases.
Fig. 11. Normalized SIF $[K_{III}/F(2/b)^{1/2}]$ versus dimensionless time $(c_{ST}/b)$, for $L/b = 2.0$, $g = 0.05$ and several values of $f$.

Fig. 12. Normalized SIF $[K_{III}/F(2/b)^{1/2}]$ versus dimensionless time $(c_{ST}/b)$, for $L/b = 3.0$, $f = 0.15$ and several values of $g$.

7. Conclusions

The diffraction of transient, viscoelastic, SH waves by a crack edge was studied in this paper. The waves were generated by a pair of equal but opposite concentrated forces, which were suddenly applied on the crack faces at a certain distance from the crack edge. Reflections of the initial and diffracted wavefronts at the lateral boundaries of the strip-like body were also taking place. Our approach provided the stress-intensity-factor at the crack tip, i.e. the intensity of the local singular stress field, as a function of time and loading/geometry/material parameters.

The loading had a particular time and space dependence, viz. it depended on a Heaviside step-function of time and a Dirac delta distribution of distance, thus the solution being Green's function of the problem. Solutions corresponding to more general loadings in time and/or space can easily be obtained by simple numerical integration (of the convolution type) of the present fundamental solution. The final numerical results reported here show that severe dynamic SIF overshoots occur. This means, of course, that the possibility of an abrupt catastrophic crack
propagation increases in the dynamic case, in comparison with the corresponding steady-state situation.

The solution method consists of a novel function-theoretic approach, which decouples two unknown complex functions contained in a single functional equation, and a numerical technique for inverting one-sided LT. In particular, the function-theoretic approach employs simple results of analytic-function theory (contour integration, Cauchy's integral and residue theorems, Cauchy's integral formula, Jordan's lemma, Abel–Tauber theorems) in the two-sided LT plane, and it can be viewed as an alternative to the standard Wiener–Hopf technique.

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