An integral equation approach to self-similar plane-elastodynamic crack problems

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Abstract. The elastodynamic problem of an expanding crack under homogeneous polynomial-form loading was reduced to the solution of a Cauchy singular integral equation. In this manner the solution of the original problem can be obtained by using well-known numerical treatments available for Cauchy SIEs. The procedure was accomplished by means of the Busemann-Chaplygin similarity technique and complex variable methods. The analysis has been restricted to the subsonic case.

1. Introduction

In treating transient elastodynamic crack motions, the similarity techniques have proved one of the most effective tools in recent years. Application of these techniques is possible when the solution of plane elastodynamic equations admits a self-similar or homogeneous form. Such a solution to a physical problem can be inferred if either the data of the problem involve a characteristic length or the only characteristic length is related to a parameter to which the solution is proportional. The principal advantage of this class of solutions is that the governing partial differential equations can be replaced by another set that contains one less independent variable than those in the original set.

The up-to-now existing similarity techniques may be classified into: a) The Smirnov-Sobolev technique [1, 2], which was utilized extensively by Kostrov [3, 4], Cherepanov and Afanas’ev [5], and Robinson and Thompson [6, 7], among others; b) Norwood’s technique [8, 9] in which integral transforms were introduced; c) Willis’ technique [10] where integral transforms were used again; d) the Busemann-Chaplygin technique [11–13], which was first utilized in solid mechanics by Miles [14] and Craggs [15] in the early sixties.

In recent years, the latter technique has been extensively utilized in crack problems by Achenbach (see e.g. [16, 17]) and Brock (see e.g. [18, 19]). Brock has largely contributed to this field by establishing some very effective

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procedures in applying the Busemann-Chaplygin technique. He has also solved numerous elastodynamic contact problems [20–23] by the same methods.

In our opinion, the Busemann-Chaplygin technique has the most simple and descriptive formulation among similarity techniques. However, the Busemann-Chaplygin technique presents some disadvantage, especially in the plane-stress/strain cases, during the final steps of the analysis where appropriate forms of complex functions are sought to accomplish the solution. This greatly obscures the elegance of the technique.

Brock ([18] and subsequent works) introduced a method to reach the solution. In the present work, we propose as an alternative a simple procedure to overcome the foregoing difficulty. By this, the original plane-elastodynamic problem is reduced to a singular integral equation with a Cauchy kernel.

The approach is rigorous and there is no possibility of losing features of the solution. After the preliminary conformal mappings, the Hilbert transform was introduced and the Keldysh-Sedov formula was utilized. The result was a Carleman integral equation whose numerical treatment, along with the additional equation resulting from the displacement continuity, is straightforward.

2. Governing equations and boundary conditions

As depicted in Fig. 1, we consider an elastic body occupying the whole space under plane-strain conditions. Assume that this body is disturbed by a rapidly expanding crack along both positive and negative x-axis at constant velocities \( v_R \) and \( v_L \), respectively.

By utilizing polar \((r, \theta)\)-coordinates, the displacements--displacement potentials and stresses--displacement potentials relations are written as

\[
\begin{align*}
\sigma_t &= \frac{\partial F_1}{\partial r} + \frac{1}{2} \frac{\partial F_2}{\partial r}, \\
\sigma_\theta &= \frac{\partial F_1}{\partial \theta} - \frac{1}{2} \frac{\partial F_2}{\partial r}, \\
\tau_r\theta &= \frac{1}{r} \frac{\partial F_1}{\partial \theta} + \frac{1}{r} \frac{\partial F_2}{\partial \theta} - \frac{1}{r^2} \frac{\partial F_2}{\partial \theta}.
\end{align*}
\]
and the equations of motion reduce to

\[ \nabla^2 F_1 = c_1^{-2} F_1, \quad \rho c_1^2 = \lambda + 2\mu, \]

\[ \nabla^2 F_2 = c_2^{-2} F_2, \quad \rho c_2^2 = \mu, \]

where a superposed dot denotes partial differentiation with respect to time, \( \rho \) is the mass density and \( \nabla^2 \) is the Laplace operator in polar coordinates.

The propagating crack is opened by self-equilibrated normal tractions along its lips. The tractions are of the fairly general form of a homogeneous polynomial in space and time. Because of the symmetry in geometry and loading with respect to the plane \( z = 0 \), the problem in the whole space may be reduced in the upper semi-space. The boundary conditions can then be written as

\[ u_\theta(x, 0, t) = 0 \text{ for } -\infty < x < -v_L t, \quad v_R t < x < \infty, \] (2.3.1)

\[ \sigma_\theta(x, 0, t) = \frac{1}{|x|} \sum_{k=0}^{n} A_k \frac{t^k}{|x|^k} \text{ for } -v_L t < x < v_R t, \quad (n \geq 1), \] (2.3.2)

\[ \tau_\theta(x, 0, t) = 0 \text{ for } -\infty < x < \infty, \] (2.3.3)
whereas the initial conditions can be stated as

\[ t \leq 0 : u^{(k)}_t, u^{(0)} = 0 \quad (k = 0, 1), \]  

(2.4)

where \( A_k \) constants and \( (\cdot)^{(k)} \) denotes the \( k \)th time derivative.

3. Dynamic similarity and conformal mappings

Following the Busemann-Chaplygin technique [11-15] we introduce a new variable \( w = r/t \) reducing thus the independent variables from \( r, \theta, t \) to \( w, \theta, t \). Moreover, differentiability of (2.3.2) and the nature of initial conditions (2.4) suggest that the problem can be stated in terms of the \((n+1)\)th time derivatives of displacements. In other words, we seek displacement-potential solutions of the form

\[ f_1 = F^{(n+1)}, \quad f_2 = F_2^{(n+1)}, \]  

(3.1)

which satisfy the equations

\[
\frac{\partial^{(n+1)} u_r}{\partial t^{(n+1)}} = \frac{\partial f_1}{\partial r} + \frac{1}{r} \frac{\partial f_2}{\partial \theta}, \quad \frac{\partial^{(n+1)} u_\theta}{\partial t^{(n+1)}} = \frac{1}{r} \frac{\partial f_1}{\partial \theta} - \frac{\partial f_2}{\partial r}.
\]  

(3.2)

Then, in the new velocity plane in Fig. 2, Equations (2.1) and (2.2) become

\[
-t \frac{\partial^n u_r}{\partial t^n} = -\frac{\partial f_1}{\partial w} - \frac{1}{w} \frac{\partial f_2}{\partial \theta},
\]  

(3.3.1)

\[
-t \frac{\partial^n u_\theta}{\partial t^n} = \frac{1}{w} \frac{\partial f_1}{\partial \theta} + \frac{\partial f_2}{\partial w},
\]  

(3.3.2)

\[
\frac{r}{\mu} \frac{\partial^n \sigma_r}{\partial t^n} = \left( 2 + \frac{w^2}{c_s^2} - \frac{2w^2}{c_1^2} \right) \frac{\partial f_1}{\partial w} - \frac{2w^2}{w} \frac{\partial f_2}{\partial \theta},
\]  

(3.3.3)

\[
\frac{r}{\mu} \frac{\partial^n \sigma_\theta}{\partial t^n} = \left( 2 - \frac{2w^2}{c_s^2} \right) \frac{\partial f_1}{\partial w} + \frac{2w^2}{w} \frac{\partial f_2}{\partial \theta},
\]  

(3.3.4)

\[
\frac{r}{\mu} \frac{\partial^n \tau_{\theta}}{\partial t^n} = -2 \frac{\partial f_1}{w} + \left( 2 - \frac{2w^2}{c_s^2} \right) \frac{\partial f_2}{\partial w},
\]  

(3.3.5)
where the relation \( \frac{\partial \theta}{\partial t} = -\left(\frac{r}{t^2}\right)\frac{\partial \theta}{\partial w} \) was used.

Due to the fact that the first dilatational wave, which is generated by the nucleation of the small flaw, propagates into an undisturbed medium, the dilatational potential \( f_1 \) must vanish at the outer wavefront in Fig. 2. The rotational potential \( f_2 \) also vanishes along the rotational wavefront since no shear change occurs outside the inner wavefront in Fig. 2. In view of the above and by combining relations (3.3) with boundary conditions (2.3), we can write

\[ f_1 = 0 \quad \text{for} \quad w = c_1, \quad \pi < \theta < 0, \]

\[ f_2 = 0 \quad \text{for} \quad w = c_2, \quad \pi < \theta < 0, \]

\[ -\frac{1}{w} \frac{\partial f_1}{\partial \theta} + \frac{\partial f_1}{\partial w} = 0 \quad \text{for} \quad -c_1 < w < -v_L, \; v_R < w < c_1 \quad \text{and} \quad \theta = 0, \pi, \quad (3.7) \]

\[ \left( 2 - \frac{w^2}{c_1^2} \right) \frac{\partial f_1}{\partial w} + \frac{2}{w} \frac{\partial f_2}{\partial \theta} = L(w) \quad \text{for} \quad -v_L < w < v_R \quad \text{and} \quad \theta = 0, \pi, \quad (3.8) \]

\[ -\frac{2}{w} \frac{\partial f_1}{\partial \theta} + \left( 2 - \frac{w^2}{c_1^2} \right) \frac{\partial f_2}{\partial w} = 0 \quad \text{for} \quad -c_1 < w < c_1 \quad \text{and} \quad \theta = 0, \pi, \quad (3.9) \]

![Fig. 2. The velocity \((w, \theta)\)-plane.](image)
where \( L(w) \) is a polynomial given by

\[
L(w) = \frac{1}{\prod_{k=0}^{n+k} (n+k)!} A_k w^{-k}.
\]

(3.10)

Moreover, combining (3.7) and (3.9) gives

\[
\frac{\partial f_1}{\partial \theta} = 0 \quad \text{for} \quad -c_1 < w < -v_L \quad \text{and} \quad v_R < w < c_1,
\]

(3.11)

\[
\frac{\partial f_2}{\partial w} = 0 \quad \text{for} \quad -c_2 < w < -v_L \quad \text{and} \quad v_R < w < c_2.
\]

(3.12)

The next step is the introduction of the Chaplygin transformation [11–13]. According to this, the area inside the half-circles \( w = c_a \) \((a = 1, 2)\) in Fig. 2 is conformally mapped onto the semi-infinite strips \( 0 < s_a < \infty, 0 < \theta < \pi \) (see for instance [14, 15]) by means of

\[
y_a = s_a + i\theta = -\cosh^{-1}(c_a/w) + i\theta \quad \text{for} \quad w \leq c_a.
\]

(3.13)

Under (3.13), Eqs. (3.4) become

\[
\frac{\partial^2 f_a}{\partial s_a^2} + \frac{\partial^2 f_a}{\partial \theta^2} = 0,
\]

(3.14)

namely the Laplace equation in the \((s_a, \theta)\)-planes. Methods of analytic-function theory are therefore applicable. In particular, we may write

\[
f_a(w, \theta) = \text{Re} \phi_a(s_a + i\theta) \quad \text{for} \quad w \leq c_a,
\]

(3.15)

where \( \phi_a \) are analytic functions of the complex variables \((s_a + i\theta)\). On the other hand, it is interesting to observe that \((\partial f_a/\partial \theta)\) may be transformed from the \((x/\ell, y/\ell)\)-plane to \((s_a, \theta)\)-planes and vice versa, simply by changing the variable \w to \s via (3.13).

We next utilize another conformal mapping in order to have the problem stated in the complex half-plane where many complex-variable results are available. The following transformation maps the semi-infinite strips \( 0 < s_a < \infty, 0 < \theta < \pi \) onto the upper half-planes \( z_a \) in Fig. 3:

\[
z_a = p_a + iq_a = \text{sech}(s_a + i\theta) = (\cosh s_a \cos \theta + i \sinh s_a \sin \theta)^{-1}.
\]

(3.16)
As will become clear soon, it is advantageous to consider the following analytic functions

\[ W_c(z_a) = \phi_c(z_a) = \frac{\partial f_a}{\partial p_a} - i \frac{\partial f_a}{\partial q_a} \]  \hspace{1cm} (3.17)

where we have utilized the Cauchy-Riemann conditions.

In order to transform now the boundary conditions from the velocity \((w, \theta)\)-plane to the \((p_a, q_a)\)-planes, we use the following relations which readily result from the transformations (3.13) and (3.16):

\[ |p_a| = w/c_a = |x|/c_a \]  \hspace{1cm} (3.18.1)

\[ c_1 p_1 = c_2 p_2, \quad p_1 = m p_2, \quad m = c_2/c_1 \]  \hspace{1cm} (3.18.2)

\[ \frac{\partial f_a}{\partial p_a} = \frac{\cosh^2 s_a}{\sinh s_a} \frac{\partial f_a}{\partial s_a} = c_a \frac{\partial f_a}{\partial w} \]  \hspace{1cm} (3.18.3)

\[ \frac{\partial f_a}{\partial q_a} = \frac{\cosh^2 s_a}{\sinh s_a} \frac{\partial f_a}{\partial \theta} = \frac{p_a (1 - p_a^2)^{1/2}}{p_a (1 - p_a^2)^{1/2}} \frac{\partial f_a}{\partial \theta} \]  \hspace{1cm} (3.18.4)
Then, Equations (3.5)-(3.12) take the following form in the half-planes:

\[
\frac{\partial f_1}{\partial p_1} = 0 \quad \text{for} \quad -\infty < p_1 < -1 \quad \text{and} \quad 1 < p_1 < \infty, \quad q_1 = 0,
\]

\[
\frac{\partial f_1}{\partial q_1} = 0 \quad \text{for} \quad -1 < p_1 < -(v_L/c_1) \quad \text{and} \quad (v_R/c_1) < p_1 < 1, \quad q_1 = 0,
\]

\[
\frac{\partial f_2}{\partial p_2} = 0 \quad \text{for} \quad -\infty < p_2 < -(v_L/c_2) \quad \text{and} \quad (v_R/c_2) < p_2 < \infty, \quad q_2 = 0.
\]

\[
\frac{2(1-m^2p_1^2)^{1/2}}{c_1} \frac{\partial f_1}{\partial q_1} + \frac{2(1-m^2p_2^2)^{1/2}}{c_2} \frac{\partial f_2}{\partial q_2} = 0,
\]

\[
\frac{2(1-m^2p_1^2)^{1/2}}{c_1} \frac{\partial f_1}{\partial p_1} + \frac{2(1-m^2p_2^2)^{1/2}}{c_2} \frac{\partial f_2}{\partial p_2} = L(p_2),
\]

where (3.22) hold for \(-(c_1/c_2) < p_a < (c_1/c_2), \quad q_a = 0\) and

\[
L(p_2) = \frac{1}{\mu} \sum_{k=0}^{\ell} \frac{(n+k)!}{k!} A_k(c_2 p_2)^{-k}.
\]

4. The singular integral equation

It is expected, since we consider only the subsonic case, that the strength of the stress singularity at the crack tips will be \(1/2\). Then, in view of (3.3) it is easy to observe that \(\phi_s(z_a)\) behaves like \([z_a + (v_L/c_a)]^{-n} - [z_a - (v_R/c_a)]^{-1/2}\). Since such strong singularities are not appropriate for our singular-integral-equation formulation, we normalize all quantities to \([z_a + (v_L/c_a)]^{-n} - [z_a - (v_R/c_a)]^{-1/2}\). In the following, quantities inside parentheses with an asterisk mean multiplication by the inverse of the latter product. Specifically for \((\partial f_1(p_1,0)/\partial p_1)\) and \((\partial f_1(p_1,0)/\partial q_1)\) we can write

\[
\left( \frac{\partial f_1}{\partial p_1} \right)^* = [p_1 + (v_L/c_1)]^n[p_1 - (v_R/c_1)]^n \left( \frac{\partial f_1}{\partial p_1} \right)
\]

\[
= m^{2n}[p_2 + (v_L/c_2)]^n[p_2 - (v_R/c_2)]^n \left( \frac{\partial f_1}{\partial p_1} \right)
\]

\[
\text{and}
\]

\[
\left( \frac{\partial f_1}{\partial q_1} \right)^* = m^{2n}[p_2 + (v_L/c_2)]^n[p_2 - (v_R/c_2)]^n \left( \frac{\partial f_1}{\partial q_1} \right)
\]

for \(q_1 = 0\).
In view of the above and (3.19)–(3.23), the mixed boundary value problem in the half-planes \((p_1, q_1)\) and \((p_2, q_2)\) can be stated as follows:

\[
\text{Re } W_\uparrow = 0 \quad \text{for } -\infty < p_1 < -1 \quad \text{and} \quad 1 < p_1 < \infty, \quad q_1 = 0, \quad (4.2)
\]

\[
\text{Im } W_\uparrow = 0 \quad \text{for } -1 < p_1 < -(v_L/c_1) \quad \text{and} \quad (v_R/c_1) < p_1 < 1, \quad q_1 = 0, \quad (4.3)
\]

\[
\text{Re } W_\uparrow = 0 \quad \text{for } -\infty < p_2 < -(v_L/c_2) \quad \text{and} \quad (v_R/c_2) < p_2 < \infty, \quad q_2 = 0, \quad (4.4)
\]

\[
\frac{2(1 - m^2 p_2^2)^{1/2}}{c_1 m^{2n}} \text{Im } W_\uparrow + \frac{(2 - p_2^2)}{c_2} \text{Re } W_\uparrow = 0, \quad (4.5.1)
\]

\[
\frac{(2 - p_2^2)}{c_1 m^{2n}} \text{Re } W_\uparrow - \frac{2(1 - p_2^2)^{1/2}}{c_2} \text{Im } W_\uparrow = L^*(p_2), \quad (4.5.2)
\]

where (4.5) hold for \(-(v_L/c_2) < p_2 < (v_R/c_2)\), \(q_2 = 0\) and

\[
L^*(p_2) = \left[ \frac{1}{\mu} \sum_{k = 0}^l \frac{(n + k)!}{k!} A_k (c_2 p_2)^{-k} \right] [p_2 + (v_L/c_2)]^n[p_2 - (v_R/c_2)]^n. \quad (4.6)
\]

Further, (4.5) are written as

\[
\text{Re } W_\uparrow = \frac{L^*(p_2) c_1 m^{2n}}{(2 - p_2^2)} + \frac{2c_1 m^{2n}(1 - p_2^2)^{1/2}}{c_2(2 - p_2^2)} \text{Im } W_\uparrow, \quad (4.7.1)
\]

\[
\text{Im } W_\uparrow = -\frac{c_1 m^{2n}(2 - p_2^2)}{2c_2(1 - m^2 p_2^2)^{1/2}} \text{Re } W_\uparrow, \quad (4.7.2)
\]

for \(-(v_L/c_1) < p_1 < (v_R/c_1)\), \(q_1 = 0\).

Then, (4.2), (4.3) and (4.7.1) form a mixed boundary value problem of the Keldysh-Sedov type with the following solution [24, 25]:

\[
W_\uparrow(p_1, 0) = (2\pi i)^{-1}(p_1 - 1)^{-1/2}[p_1 - (v_R/c_1)]^{-1/2}[(v_L/c_1)]^{-1/2}[p_1 + (v_L/c_1)]^{-1/2}
\]

\[
\times \left[ T + (v_L/c_1)]^{1/2}(T - p_1)^{-1} \right. dT
\]

\[
+ (Ap_1^2 + Bp_1 + C)(p_1^2 - 1)^{-1/2}
\]

\[
\times [p_1 - (v_R/c_1)]^{-1/2}[p_1 + (v_L/c_1)]^{-1/2}, \quad (4.8)
\]
where \( A, B \) and \( C \) are imaginary constants. A careful inspection of the latter expression reveals that

\[
\text{Re} \ W^\star\!(p_1, 0) = 0, \quad \text{Im} \ W^\star\!(p_1, 0) = i \text{ Im} \ W^\dagger\!(p_1, 0),
\]

for \( -(v_L/c) < p_1 < (v_R/c). \) \hfill (4.9)

In view of the above result, (4.5.2) or (4.7.1) becomes

\[
\text{Im} \ W^\star_{2^*} = -\frac{c_2 L^\star\!(p_2)}{2(1 - p_2^2)^{1/2}} \quad \text{for} \quad -(v_L/c) < p_2 < (v_R/c), \quad q_2 = 0,
\]

\hfill (4.10)

whereas from (4.7.2) it is valid that

\[
\text{Re} \ W^\star_{2^*} = \frac{2(1 - m^2 p_2^2)^{1/2}}{m^2 - 1(2 - p_2^2)} \quad \text{Im} \ W^\dagger\! \quad \text{for} \quad -(v_L/c) < p_2 < (v_R/c), \quad q_2 = 0.
\]

\hfill (4.11)

At this point we introduce the Hilbert transform which is also known as the dispersion relation or the Schwarz integral formula \([26-28]\). According to this

\[
\text{Re} G(p) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\text{Im} \ G(T)}{T - p} \, dT, \quad \text{Im} \ G(p) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\text{Re} \ G(T)}{T - p} \, dT.
\]

\hfill (4.12)

Then, the second of (4.12) along with (4.4) and (4.11) gives

\[
\text{Im} \ W^\star_{2^*}(p_2, 0) = -\frac{2}{\pi m^2 - 1} \int_{-(v_L/c)}^{(v_R/c)} \frac{(1 - m^2 T^2)^{1/2}(\text{Im} \ W^\dagger\!(T))}{(2 - T^2)(T - p_2)} \, dT.
\]

\hfill (4.13)

In this way, we are able to construct a singular integral equation with a Cauchy kernel for the unknown function \( \text{Im} \ W^\star_{2^*}(p_2, 0), - (v_L/c) < p_2 < (v_R/c), \) by combining (4.10) and (4.13)

\[
\int_{-(v_L/c)}^{(v_R/c)} \frac{(1 - m^2 T^2)^{1/2}(\text{Im} \ W^\dagger\!(T))}{(2 - T^2)(T - p_2)} \, dT = -\frac{\pi c_2 m^{2n-1} L^\star\!(p_2)}{4(1 - p_2^2)^{1/2}},
\]

for \( -(v_L/c) < p_2 < (v_R/c). \) \hfill (4.14)

The above equation must be supplemented by another integral equation which characterizes the behavior of the unknown function \( \text{Im} \ W^\star_{2^*}(p_2) \) at the end points \((v_R/c)\) and \( -(v_L/c)\). For square root singularities in stresses at the crack tips the single-valuedness condition must be utilized. This states that
the following relation holds in the velocity plane [29]:

\[ \int_{-v_L}^{v_R} J(T) \, dT = 0, \quad J(w) = \frac{\partial}{\partial w} [u_0(w, y = 0^+) - u_0(w, y = 0^-)]. \] (4.15)

Thus, in view of relations (3.3.2), (3.9), (3.17) and (4.1.2) and condition (4.15), the integral equation complementary to (4.14) is

\[ \int_{-(v_L/c_2)}^{(v_R/c_2)} \frac{T(1 - m^2 T^2)^{1/2}}{(2 - T^2)} (\text{Im} \, W^*(T)) \, dT = 0. \] (4.16)

We can observe in (4.16) that this condition consists of specifying the first moment ([30], p. 1048) of the function \([1 - m^2 T^2]^{1/2} (2 - T^2) [\text{Im} \, W^*(T)]\) over the interval \([- (v_L/c_2), (v_R/c_2)]\) as in the case of the crack problem for a semi-infinite solid with heated bounding surface [31].

Now the set (4.14) and (4.16) can be solved numerically. In recent years an expanding number of methods has appeared for doing this (see for instance [30–37]). Perhaps the simplest but still very effective technique is the one introduced by Erdogan and Gupta [34], supplemented by Krenk's interpolation [36]: First we change the interval \([- (v_L/c_2), (v_R/c_2)]\) into \([-1, 1]\) by writing

\[ \int_{-v_L/c_2}^{v_R/c_2} g(T) \, dT = \left( \frac{v_R + v_L}{2c_2} \right) \int_{-1}^{1} g[T(S)] \, dS, \] (4.17)

\[ T = T(S) = \frac{1}{2} \left[ \left( \frac{v_R - v_L}{c_2} \right) + \left( \frac{v_R + v_L}{c_2} \right) S \right]. \]

Then, the set (4.14) and (4.16) is written as

\[ \int_{-1}^{1} \frac{R(S)X(S)}{S - p} \, dS = \pi M(p), \quad -1 < p < 1, \] (4.18.1)

\[ \int_{-1}^{1} [(v_R + v_L)S + (v_R - v_L)] R(S) X(S) \, dS = 0, \] (4.18.2)

where

\[ R(S) = \frac{(1 - m^2 T^2)^{1/2}}{(2 - T^2)}, \quad X(S) = \text{Im} \, W^*(T), \] (4.19)

\[ M(p) = -c_2 m^{2\alpha - 1} L^*(p_2), \quad p_2 = \frac{1}{2} \left[ \left( \frac{v_R - v_L}{c_2} \right) + \left( \frac{v_R + v_L}{c_2} \right) p \right]. \]
According to the Erdogan-Gupta quadrature the solution of (4.18) is given by the following system of \( h \) linear equations to determine \( Y(S_1), \ldots, Y(S_h) \):

\[
\sum_{j=1}^{h} \frac{1}{h} R(S_j) Y(S_j) = \frac{1}{S_j - p_u}, \quad (u = 1, \ldots, h - 1),
\]

\[
\sum_{j=1}^{h} \left[ (v_R + v_L) S_j + (v_R - v_L) \right] R(S_j) Y(S_j) = 0, \quad S_j = \cos \left( \frac{\pi}{2h} (2j - 1) \right), \quad p_u = \cos \left( \frac{\pi u}{h} \right),
\]

and

\[
X(S) = \frac{Y(S)}{(1 - S^2)^{1/2}},
\]

where the intensity values \( Y(1), Y(-1) \) at the crack tips can be found by

\[
Y(1) = \frac{1}{h} \sum_{j=1}^{h} \frac{\sin \left[ \frac{2h - 1}{4h} (2j - 1) \pi \right]}{\sin \left[ \frac{2j - 1}{4h} \pi \right]} Y(S_j),
\]

\[
Y(-1) = \frac{1}{h} \sum_{j=1}^{h} \frac{\sin \left[ \frac{2h - 1}{4h} (2j - 1) \pi \right]}{\sin \left[ \frac{2j - 1}{4h} \pi \right]} Y(S_{h+1-j}).
\]

To determine the \( \sigma_\theta(r, 0, t) \)-stress ahead of the moving crack tips, it is seen from (3.3.4) that the values of the Re \( W^*_t \) and Im \( W^*_t \) functions are needed in the ranges \( -1 < p_1 < -(v_L/c_2) \) and \( (v_R/c_2) < p_1 < 1 \).

Regarding Im \( W^*_t \) we utilize (4.13) after obtaining the values of Im \( W^*_t \) in \( \left[ -(v_L/c_2), (v_R/c_2) \right] \) by the foregoing numerical procedure.

Now for Re \( W^*_t \) we consider again the following Keldysh-Sedov problem in the \( z_1 \)-plane:

\[
\text{Re } W^*_t(p_1, 0) = 0
\]

\[
\text{for } -\infty < p_1 < -1, \quad -(v_L/c_1) < p_1 < (v_R/c_1), \quad 1 < p_1 < \infty,
\]

\[
\text{Im } W^*_t(p_1, 0) = 0 \quad \text{for } -1 < p_1 < -(v_L/c_1), \quad (v_R/c_1) < p_1 < 1,
\]

which has the solution [24, 25]

\[
W^*_t = \frac{A p_1^2 + B p_1 + C}{(p_1^2 - 1)^{1/2} \left[ p_1 - (v_R/c_1) \right]^{1/2} \left[ p_1 + (v_L/c_1) \right]^{1/2}},
\]
where $A$, $B$ and $C$ are imaginary constants. Since solution (4.24) is valid everywhere the as yet unknown constants can be determined by equating the values of the numerically found $\text{Im } W^\dagger$ (Equations (4.20) and (4.21)) with the analytical result (4.24) at points inside the interval $[-(v_L/c_1), (v_R/c_1)]$. In the ranges of interest $[-1, -(v_L/c_1)]$ and $[(v_R/c_1), 1]$ the $W^\dagger(p_1, 0)$ in (4.24) takes only real values and we thus have the complete solution.

The normal stress on the crack line, ahead of the moving crack tips, can then be easily obtained by transforming the values of $\text{Re } W^\dagger(p_1, \theta)$ and $\text{Im } W^\dagger(p_2, \theta)$ back to the velocity plane through (3.18) and by using (3.3.4). For the stress intensity factor the interpolation values (4.22) must also be used.

5. Conclusions

In this paper we have presented a novel procedure to analyze the problem of a rapidly propagating crack under a plane-strain general loading. The usual assumptions of linear elasticity were considered and the method of dynamic similarity was employed in order to introduce complex-variable methods.

A singular integral equation was finally constructed and an algorithm for its numerical solution was also given. We believe that the whole procedure is straightforward and one may easily take information about stresses on the crack line and their intensities for each particular problem by using the results of the paper.

References

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