FINITE LENGTH CRACK MOVING IN A VISCOELASTIC STRIP UNDER IMPACT—I. THEORY

H. G. GEORGIADIS†
33–35 G. Papandreou St., Athens 16231, Greece

Abstract—A dynamic crack problem of the transient type was considered within the context of the linear theory of viscoelasticity. This involves a finite-length crack moving in a strip-like viscoelastic body under impact loading. Laplace and Fourier transforms were utilized and the resulting dual integral equations were reduced to a Fredholm integral equation of the second kind.

INTRODUCTION

In recent years there is a need for fracture mechanics to obtain solutions of non-idealized crack problems. This results as a consequence of the broad application of fracture-mechanics concepts in a wide class of materials which may have also special geometrical configurations. Such a non-idealized situation may involve both some more general constitutive relations for the material response and finite boundaries of the cracked body. Thus, the mathematical crack problems become more and more difficult and the techniques for their solution more and more sophisticated.

One may find two works in literature somewhat related to the present study. Chen and Sih[1] analyzed the problem of an elastic strip containing a stationary central crack under impact loading whereas Atkinson and Popelar[2] solved the problem of a viscoelastic strip containing a dynamic semi-infinite crack under impact loading, too.

In the present paper, we have considered a fast running crack in a strip of a viscoelastic material under anti-plane impact loading. The crack was of finite length and of constant velocity. Linear viscoelastic theory was utilized throughout the analysis. The solution was effected by means of integral transforms and a dual integral equation method.

Theoretical investigations in fast fracture of viscoelastic materials was originated by Willis[3] and continued in recent years by Atkinson and List[4], Atkinson and Popelar[2] and Walton[5]. The present work belongs to the context of these studies dealing however with a somewhat more difficult problem.

BASIC RELATIONS AND BOUNDARY CONDITIONS

Assume that a linearly viscoelastic, isotropic and homogeneous body, in the form of an infinite strip having infinite thickness and containing a central constant-length crack, is loaded by impact anti-plane displacements acting on the strip faces perpendicularly to the (x, y)-plane of the cross-section. The crack moves at a constant velocity \(v\) along the middle distance of the strip height \(2h\) and its length is \(2a\). It is well known, this type of deformation implies the existence only of the out-of-plane \(\tau_{xz}\) and \(\tau_{yz}\) stresses and the \(w\) displacement. Then, in respect to a \((x', y')\) stationary Cartesian coordinate system the governing equations are written as [6, 7]

\[
\tau_{xz} = \frac{Q_1(D)}{P_1(D)} \frac{\partial w}{\partial x'}, \quad \tau_{yz} = \frac{Q_1(D)}{P_1(D)} \frac{\partial w}{\partial y'},
\]

\[
\frac{\partial \tau_{xz}}{\partial x'} + \frac{\partial \tau_{yz}}{\partial y'} = \rho \frac{\partial^2 w}{\partial t^2},
\]

where \(\rho\) is the mass density of the viscoelastic medium, \(t\) the time and \(D\) the time-derivative.

†From September 1987 at MEAM, The University of Michigan, Ann Arbor, MI 48109, U.S.A.
operator defined as

\[ Df = \frac{\partial f}{\partial t} \]  \hspace{1cm} (3)

The above differential operator law formulation is more suitable in this study than the usual one involving convolutions of strains or stresses and relaxation or creep functions, respectively. For instance, the differential operators for the standard linear solid read as

\[ \frac{Q_i(D)}{P_i(D)} = \frac{\mu_\alpha \left( \frac{d}{dt} + \frac{1}{\tau} \right)}{\frac{d}{dt} + \frac{1 + f}{\tau}} \]  \hspace{1cm} (4)

where \( \mu_\alpha, f \) and \( \tau \) are independent material constants.

Introducing now a moving coordinate system \( x = x' - vt, y = y' \) allows us to write (2) as

\[ \left( 1 - \frac{v^2 \rho P_i(D)}{Q_i(D)} \right) \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \rho P_i(D) \left( -2v \frac{\partial^2 w}{\partial x \partial t} + \frac{\partial^2 w}{\partial t^2} \right). \]  \hspace{1cm} (5)

It is noted that (1) is still valid in the new system dropping out the primes.

The boundary conditions of the problem can be written as

\[ w(x, \pm h, t) = \pm w_0 H(t) \text{ for } -\infty < x < \infty \]  \hspace{1cm} (6a)

\[ w(x, 0, t) = 0 \text{ for } |x| > a \]  \hspace{1cm} (6b)

\[ \tau_{\alpha}(x, 0, t) = 0 \text{ for } |x| < a \]  \hspace{1cm} (6c)

where \( H(t) \) the Heaviside step function. Previous experience in similar problems [2, 8, 9] implies that the above form of the boundary conditions is not convenient to analyze the crack problem. To this end, we consider the following auxiliary problem and then by superposition we may obtain solving the original one

\[ w(x, \pm h, t) = 0 \text{ for } -\infty < x < \infty \]  \hspace{1cm} (7a)

\[ w(x, 0, t) = 0 \text{ for } |x| > a \]  \hspace{1cm} (7b)

\[ \tau_{\alpha}(x, 0^+, t) = -\frac{Q_i(D)}{P_i(D)} \frac{w_0}{h} H(t) \text{ for } |x| < a. \]  \hspace{1cm} (7c)

However, since in this study we are mainly interested to determine stress intensity factors, solving problem (7) suffices for our purpose.

**ANALYSIS**

As is already mentioned the solution was effected by means of integral transforms. The Laplace transform over time and the Fourier exponential and cosine transforms over space were utilized. Following the definitions in [10] we take

(a) The Laplace transform pair

\[ f_i(p) = \mathcal{F}[f(t)] = \int_0^\infty f(t) e^{-pt} dt, \quad f(t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} f(p) e^{pt} dp. \]  \hspace{1cm} (8a,b)
(b) The Fourier exponential transform pair

\[ f_{FE}(s) = f^*(s) = \int_{-\infty}^{\infty} f(x) e^{-isx} dx, \quad f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f_{FE}(s) e^{isx} ds. \]  

(9a,b)

(c) The Fourier cosine transform pair

\[ f_{FC}(s) = \hat{f}(s) = \int_{0}^{\infty} f(x) \cos (sx) dx, \quad f(x) = \frac{2}{\pi} \int_{0}^{\infty} f_{FC}(s) \cos (sx) ds. \]  

(10a,b)

Under (8a) and quiescent initial conditions, relation (5) becomes

\[ \left( 1 - \frac{v^2}{c^2} \right) \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{1}{c^2} \left( -2v \rho \frac{\partial w}{\partial x} + \rho^2 w \right) \]  

(11)

and further via (9a) it is transformed to

\[ \frac{\partial^2 w^*}{\partial y^2} - \left[ s^2 + \frac{(p + ivs)^2}{c^2} \right] w^* = 0 \]  

(12)

where

\[ c^2 = [\tilde{c}(p)]^2 = \frac{\tilde{\mu}(p)}{\rho} = \frac{1}{\rho} \tilde{Q}_1(p) \]  

(13)

in relation (11), and

\[ c^2 = [\tilde{c}^*(p + ivs)]^2 = \frac{\tilde{\mu}^*(p + ivs)}{\rho} = \frac{1}{\rho} \tilde{Q}_1^*(p + ivs) \]  

(14)

in relation (12). For instance, by considering again the standard linear solid, see (4), its transformed 'shear modulus' is written as

\[ \bar{\mu}^* = \frac{\mu_0 \left[ (p + ivs) + \frac{1}{\tau} \right]}{(p + ivs) + \frac{1 + \frac{1}{\tau}}{\tau}}. \]  

(15)

The \( \tau_\gamma \)-stress, see (1b), which enters into the boundary conditions of the problem after the successive transformations over time and space becomes [6]

\[ \bar{\tau}_\gamma^*(s, y, p) = \tilde{\mu}^*(p + ivs) \frac{\partial}{\partial y} \tilde{w}^*(s, y, p). \]  

(16)

Obviously, (12) can be regarded as an ordinary differential equation for \( \tilde{w}(s, y, p) \) considered as a function of \( y \) and hence one may write

\[ \tilde{w}^*(s, y, p) = C(s, p) \sinh \left[ \gamma(h - y) \right]. \]  

(17)

taking into account (7) and the anti-symmetry property \( w(x, y, t) = -w(x, -y, t) \). In (17) \( C(s, \)
\( p \) is an unknown function to be determined and \( \gamma \) is defined by

\[
\gamma = \left[ s^2 + \frac{(p + ivs)^2}{c^2(p + ivs)} \right].
\] (18)

Note now the relationship between the exponential and cosine Fourier transforms [10]

\[
C[f(x)] = \frac{1}{2} E_s[f(|x|)]
\]

where \( C[f(x)] = f_{fc} \) and \( E_s[f(x)] = f_{fe} \). Therefore, it is allowable to write (12) as

\[
\frac{\partial^2 w}{\partial y^2} - \left[ s^2 + \frac{(p + ivs)^2}{c^2} \right] \frac{\partial}{\partial s} w = 0.
\] (20)

As a consequence of the above and taking into account (10b) and (17), \( \tilde{w} \) admits the representation

\[
\tilde{w}(x, y, p) = 2 \int_0^\infty C(s, p) \sinh \left( \gamma(h - y) \right) \cos (sx) \, dx.
\] (21)

It is noticeable that one may not arrive directly at (20) from (11) via (10a) because of the inconvenience of the Fourier cosine transform to apply to the derivatives with respect to \( x \) in eq. (11).

Combining now the boundary condition (7b) and (7c) with (16) and (21) the solution of the original problem for \( y = 0 \) is reduced to that of the dual integral equations

\[
\int_0^\infty C(s, p) \sinh (\gamma h) \cos (sx) \, ds = 0 \quad \text{for } |x| > \alpha
\] (22a)

\[
\int_0^\infty \gamma C(s, p) \cosh (\gamma h) \cos (sx) \, ds = \frac{\pi w_0}{2ph} \quad \text{for } |x| < \alpha.
\] (22b)

By setting further

\[
A(s, p) = C(s, p) \sinh (\gamma h)
\]
(23)

(22) may be rewritten as

\[
\int_0^\infty A(s, p) \cos (sx) \, ds = 0, \quad |x| > \alpha
\] (24a)

\[
\int_0^\infty s \, A(s, p) \frac{\gamma}{s} \cosh (\gamma h) \cos (sx) \, ds = \frac{\pi w_0}{2ph}, \quad |x| < \alpha.
\] (24b)

For the solution of dual integral equations with trigonometrical kernels such as (24) much work has been done in the past by Sih and co-workers (see e.g. refs [1] and [11]). An illuminated account is also given by Embley[12]. These researchers followed Copson's method[13] and utilized the procedure in numerous crack problems.
Following thus the work in [11] and [12] one may find the following expression for the as yet unknown $A(s, \eta)$-function

\[ A(s, \eta) = \frac{\pi w_0 a^2}{2 \beta_0 p s} \left[ \varphi(\xi, p) J_0(sa\xi) \right] d\xi \]  

where

\[ \varphi(\xi, p) + \int_0^1 K(\xi, \eta, p) \varphi(\eta, p) \, d\eta = \xi^{-1}, \]  

\[ K(\xi, \eta, p) = \frac{(\xi \eta)^{\frac{\gamma \eta}{b_0}}} {b_0} \left[ \frac{\gamma}{s} \coth(\gamma h) - b_0 \right] J_0(s \xi) J_0(s \eta) \, ds, \]  

\[ b_0 = \lim_{\gamma \to \infty} \left[ \frac{\gamma}{s} \coth(\gamma h) \right] = \left[ 1 - \frac{\nu^2}{c^2(\infty)} \right] = \beta_\infty, \]  

\[ (x/a) < \xi < 1, \quad |x| < a \]  

(29a)

\[ \eta = \xi/a, \quad 0 < \xi < a, \quad 0 < \eta < 1. \]  

(29b)

In (28) $c(\infty) = c_\infty = (\mu_\infty / \rho)^{\frac{1}{2}}$ is the short-time wave speed.

Furthermore, integrating by parts in (25) we obtain

\[ A(s, \eta) = \frac{\pi w_0 a}{2 \beta_0 p s} \left[ \varphi(\xi, p) J_1(sa\xi) - \int_0^1 \frac{d}{d\xi} \left[ (\xi^{-1} \varphi(\xi, p)) J_1(sa\xi) \right] d\xi \right]. \]  

(30)

In the above relations $J_0(\cdot)$ and $J_1(\cdot)$ stand for the Bessel functions of the first kind of order zero and one.

Since we are interested in the stress intensity at the crack tip, we take $\xi = 1$. In addition only the singular portion of $A(s, \eta)$ as is given in (30) is taken into account. Hence, we get

\[ A(s, \eta) \approx \frac{\pi w_0 a}{2 \beta_\infty p s} \varphi(1, p) J_1(sa) \]  

(31)

where $\varphi(1, p)$ results as the solution of the Fredholm integral equation

\[ \varphi(\xi, p) + \int_0^1 K(\xi, \eta, p) \varphi(\eta, p) \, d\eta = \xi^{-1} \varphi \]  

(32)

and the kernel in the latter is given by

\[ K(\xi, \eta, p) = \frac{\xi \eta^i}{\beta_\infty} \left[ \frac{\gamma}{s} \coth(\gamma h) - \beta_\infty \right] J_0(s) J_0(s \eta) \, ds. \]  

(33)

In view of (31) and for large values of $s$ (which correspond to small values of $x$, namely near crack tips) $(\partial \tilde{w} / \partial y)$ may be expressed as
\[
\frac{\partial \bar{w}}{\partial y} = \frac{\varphi(1, p)}{p} \frac{w_0 a}{h} \int_{0}^{\infty} J_1(sa) \cos(sx) \, dx.
\]  

(34)

Furthermore, by using standard results in [14] as regards the integral in (34) the latter relation becomes

\[
\frac{\partial \bar{w}}{\partial y} = \frac{\varphi(1, p)}{p} \frac{w_0 a^2}{h} \left[ \frac{1}{(x^2 - a^2)^\frac{3}{2}} \right] [x + (x^2 - a^2)^\frac{1}{2}]^\frac{1}{2}.
\]

(35)

Now, the singular portion of the function of interest, i.e. of the \( \tau_{yz}(x, 0, t) \)-stress, can be written in the Laplace transform plane with the help of (1b) and (35)

\[
\bar{\tau}_{yz}(x, 0, p) = \bar{\mu}(p) \frac{\varphi(1, p)}{p} \frac{w_0 a^2}{h} \left[ \frac{1}{(x^2 - a^2)^\frac{3}{2}} \right] [x + (x^2 - a^2)^\frac{1}{2}].
\]

(36)

Then, it is easy to extract the stress intensity factor at the leading tip of the crack by using (36) and the well-known relation

\[
k(t) = \lim_{x \to \alpha^+} \left[ 2(x - \alpha)^\frac{3}{2} \tau_{yz}(x, 0, t) \right].
\]

(37)

Thus, the stress intensity factor in the Laplace transform plane is given by

\[
k(p) = \bar{\mu}(p) \frac{\varphi(1, p)}{p} \frac{w_0 a^2}{h}.
\]

(38)

Finally, the time dependent stress intensity factor in our viscoelastic dynamic crack problem is given by

\[
k(t) = \frac{w_0 a^2}{h} L^{-1} \left[ \bar{\mu}(p) \frac{\varphi(1, p)}{p} \right].
\]

(39)

CONCLUDING REMARKS

In this work the general procedure of analyzing dynamic crack problems in linear viscoelastic bodies of finite dimensions was given. The solution of the anti-plane shear case was derived but the method is also applicable to the more complicated plane-strain case.

To extract numerical results a sufficient usage of methods of Numerical Analysis is required. It is hoped that numerical results for some simple viscoelastic models will appear in a future paper. However, it seems that some remarks on this subject are in order: Firstly, a numerical treatment of the integral in (33) is required. This can be done by normalizing the semi-infinite interval of integration to \([-1, 1]\) and then by using an integration rule. The Fredholm integral equation (32) can either treat numerically or by the method of successive approximations. Finally, the Laplace transform inversion in (39) can be accomplished by using the well-known methods of Papoulis[15] or Miller and Guy[16].

REFERENCES


(Received 6 November 1986)